Strongly σ-Complete Boolean Algebras: A Unicorn?

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The collection clop(X) of all closed-open sets of any topological space X is a Boolean algebra under finite union and intersection, and any Boolean algebra B is clop(X) in several zero-dimensional topological spaces X, where "zero-dimensional" (ZD) means that X is Hausdorff and clop(X) is a base for the topology. When $B = \text{clop}(X)$ is a complete Boolean algebra, the ZD space X has the property that each open set is a dense subset of some element of B, and conversely. These spaces are called "extremally disconnected". But what if $B = \text{clop}(X)$ is only σ-complete (closed for countable meets and joins)? We know that for some classes of ZD X, every open set of the form $\text{Coz}(f) = x : |f(x)| > 0$ for continuous $f : X \to R$ is a dense subset of some element of B. Spaces with this property are called "basically disconnected". This motivates the following.

Definition: A Boolean algebra B is "strongly σ-complete" if every zero-dimensional space X with clop(X) isomorphic to B is basically disconnected. We easily show that for a Boolean algebra B, complete $\implies$ strongly σ-complete $\implies$ σ-complete.

A strongly σ-complete Boolean algebra is a unicorn, in the sense that every σ-complete B might be strongly σ-complete, in which case the definition would be vacuous. And we have not found one which is not complete. And what could possibly be an algebraic characterization of the strongly σ-complete Boolean algebras?

This talk describes some aspects of these questions, but does not find a unicorn.

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