On Pseudovarieties Satisfying $\mathsf{V} \textcircled{m} \mathsf{G} = \mathsf{EV}$

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By a pseudovariety we mean a class of finite semigroups that is closed under homomorphic images, subsemigroups and finite products. A relational morphism between semigroups S and T is a relation $S \to T$ with domain S which, as a subset of $S \times T$, is a subsemigroup. The group kernel $K(S)$ of S is the intersection of the subsemigroups $\tau^{-1}(1)$ over all relational morphisms $\tau\colon S \to G$ into finite groups. For finite semigroups, $K(S)$ is a subsemigroup containing the idempotents that is closed under weak conjugation: if s' is a weak inverse of s, that is, if $s'ss' = s'$, then $sK(S)s' \cup s'K(S)s \subseteq K(S)$. Rhodes' Type II Conjecture states that, in case S is finite, the group kernel $K(S)$ is the smallest subsemigroup of S containing the idempotents and closed under weak conjugation. The Mal'cev product of two pseudovarieties V and W is defined as follows:

$$\mathsf{V} \textcircled{m} \mathsf{W} = \{M \mid \text{there is a relational morphism } \tau\colon M \to N \text{ with } N \in \mathsf{W} \text{ and such that } \tau^{-1}(e) \in \mathsf{V} \text{ for every idempotent } e \in N\}.$$

Let S be a finite semigroup and V be a pseudovariety. We have $S \in \mathsf{V} \textcircled{m} \mathsf{G}$ if and only if $K(S) \in \mathsf{V}$ where G is the pseudovariety of all finite groups. In addition, $S \in \mathsf{EV}$ if and only if the subsemigroup $\langle E(S) \rangle$ generated by the idempotents belongs to V. Hence, the inclusion $\mathsf{V} \textcircled{m} \mathsf{G} \subseteq \mathsf{EV}$ always holds. The purpose of this presentation is to discuss the results of a study that focused on cases when the pseudovariety equality

$$\mathsf{V} \textcircled{m} \mathsf{G} = \mathsf{EV} \tag{1}$$

holds.

A related problem, which has also been investigated is to consider instead the equation

$$\mathsf{V} * \mathsf{G} = \mathsf{EV}, \tag{2}$$

where the star denotes semidirect product of pseudovarieties. There are connections between the two problems since $\mathsf{V} * \mathsf{G}$ is always contained in $\mathsf{V} \textcircled{m} \mathsf{G}$, with equality holding whenever (but not exclusively) V is local. Thus, for a pseudovariety V to be a solution of the equation (1) is a weaker property than to be a solution of (2). For instance, while the fact that the pseudovariety J (of all finite $\mathcal{J}$-trivial semigroups), which is not local, satisfies (1) is not very difficult, it is a much deeper fact that J also satisfies (2). On the other hand, for the pseudovarieties DS and DA, of all finite semigroups in which all regular elements respectively lie in groups or are idempotents, are both local and solutions of both equations. A whole interval of solutions of both equations is $[\mathsf{Sl}, \mathsf{Com}]$ where Sl and Com are the pseudovarieties of all finite, respectively, semilattices and commutative semigroups. Indeed, by a theorem of Ash, Sl is a solution of (2) and, clearly, $\mathsf{ESl} = \mathsf{ECom}$. While the pseudovariety Sl is local, Com is not. During the presentation, we will derive pseudoidentities that can function as either solutions of (1) for all pseudovarieties that adhere to them or as counterexamples. Initially, we were motivated to investigate the pseudovariety equation (1) for a pseudovariety

V satisfying $\mathsf{SI} \subseteq \mathsf{V} \subseteq \mathsf{DA}$. However, we later found out that there are subpseudovarieties within J, situated between SI and DA, which do not satisfy the pseudovariety equation (1). We provide a set of semigroups where a pseudovariety includes at least one of the semigroups mentioned in the set if and only if the pseudovariety comprises a subpseudovariety that does not satisfy the pseudovariety equation (1). Moreover, a pseudovariety does not include any semigroups of the set if and only if the pseudovariety is in EB where $\mathsf{B} = [\![x^2 = x]\!]$.

Acknowledgment:
The authors were partially supported by CMUP, member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020. The second author also acknowledges FCT support through a contract based on the "Lei do Emprego Científico" (DL 57/2016).

Keywords: semigroup, pseudovariety, Mal'cev product, relational morphism, idempotent-generated subsemigroup