MATH RESEARCH AT UNC CHARLOTTE 2023

Project 1: Numerical solution to a coefficient inverse problem for parabolic equations

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Project description. The main aim of this project is to propose a numerical method to solve a nonlinear coefficient inverse problem for nonlinear parabolic equations. Let $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $d \geq 2$, be a matrix-valued function that represents the thermal conductivity of an anisotropic medium. Assume that A belongs to the class C^2 and that there exist two positive numbers λ_1 and λ_2 such that

$$\lambda_1|\xi|^2 \leq A(\mathbf{x})\xi \cdot \xi \leq \lambda_2|\xi|^2 \quad \text{for all } \mathbf{x}, \xi \in \mathbb{R}^d.$$

Let $f : [0,\infty) \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a function in the class C^1. Consider the quasi-linear parabolic equation

$$\left\{\begin{array}{rcll} u_t(\mathbf{x},t) &=& \mathrm{div}(A(\mathbf{x})\nabla u(\mathbf{x},t)) + c(\mathbf{x})f(t,\mathbf{x},u(\mathbf{x},t),\nabla u(\mathbf{x},t)) & (\mathbf{x},t) \in \mathbb{R}^d \times (0,\infty),\\ u(\mathbf{x},0) &=& u_0(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d. \end{array}\right.$$

We are interested in the following problem.

Problem 1 (Coefficient inverse problem for quasi-linear parabolic equation)**.** *Fix $T > 0$. Determine the function $c(\mathbf{x})$ from the boundary measurements*

$$F(\mathbf{x},t) = u(\mathbf{x},t) \quad \textit{and} \quad G(\mathbf{x},t) = A(\mathbf{x})\nabla u(\mathbf{x},t)\cdot \nu(\mathbf{x})$$

for all $(\mathbf{x},t) \in \partial\Omega \times [0,T]$.

Problem 1 is severely ill-posed in the sense that small noise in data may cause huge errors in the computed solutions. Although numerically solving this problem is challenging, we strongly believe that our proposed method, based on Carleman estimates and the contraction principle, will deliver reliable solutions even when the given data are highly noisy.

Problem 1 has uncountable practical applications. In fact, assume that the region Ω is not accessible. By measuring some boundary information of the function $u(\mathbf{x},t)$ and the flux $A(\mathbf{x})\nabla u(\mathbf{x},t)\cdot\nu$, $(\mathbf{x},t) \in \partial\Omega \times (0,T)$, one can identify the internal information of the coefficient $c(\mathbf{x})$, $\mathbf{x} \in \Omega$. This enables us to inspect Ω without destructing it. We name here two examples: determination of the spatially distributed temperature inside a solid from the boundary measurement of the heat and heat flux in the time domain [5], effective monitoring the heat conduction processes in steel industries, glass and polymer-forming, and nuclear power station [6]. We recall here another specific example of bioheat transfer. When $f(t,\mathbf{x},u(\mathbf{x},t),\nabla u(\mathbf{x},t)) = u(\mathbf{x},t)$, the coefficient $c(\mathbf{x})$ represents the blood perfusion. The knowledge of this coefficient plays a crucial role in calculating the temperature of the blood flowing through the tissue, see [4]. Many publications in this area [1, 2, 3, 4, 7, 8, 9] employed the least-squares optimization with some Tikhonov regularization terms. This method is widely used in the scientific community but it has a drawback. Due to the nonlinearity, the optimization method might not be able to deliver reliable solutions unless a good initial guess is given. Our goal is to solve Problem 1 without requesting an initial guess. Success is guaranteed by using Carleman estimates and the contraction principle.

REU Students' role. We welcome students with a background in differential equations and linear algebra to conduct research on this project. Students will write parts of the computational codes for the proposed algorithms. On the other hand, students will be trained in the theory of differential equations and inverse problems. They will prove some parts of the theorems guaranteeing the efficiency of the proposed method. In addition, students will serve as testers. More precisely, they will run our computational codes for a variety of values of some parameters which influence the data. The goal of these tests will be to find the optimal values of those parameters.

References

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