MATH RESEARCH AT UNC CHARLOTTE 2023

Project 4: On the zeros of partial sums of the Riemann zeta function.

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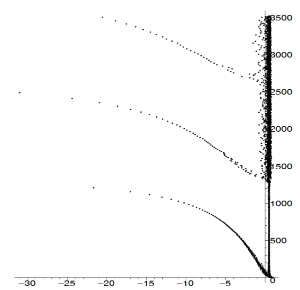
Project description. Prime numbers are the building blocks of modern cyber security. The Riemann zeta function $\zeta(s)$ is defined by $\zeta(s) = 1+2^{-s}+3^{-s}+\ldots$, where s is a complex number with $\mathrm{Re}(s) > 1$. Zeros of $\zeta(s)$, which encode valuable information about the prime numbers, is a central object of study in number theory. One approach toward understanding the location of zeros of $\zeta(s)$ involves the study of partial sums of $\zeta(s)$, defined by the Dirichlet polynomials $\zeta_N(s) = 1+2^{-s}+\cdots+N^{-s}$. An interesting question is the logical relation between the existence of zeros of $\zeta_N(s)$ and of $\zeta(s)$ in certain regions. The study of partial sums in this context was initiated by Turán [10], who showed that if $\zeta_N(s)$ has no zeros in some half-plane then the celebrated Riemann hypothesis is true. It was only in 1983 that the hypotheses required above by Turán on the zeros of $\zeta_N(s)$ were shown to be false by Montgomery [6]. Since then, zeros of $\zeta_N(s)$ and related questions have been studied widely by, among others, see e.g. [1, 2, 3, 11, 5, 4, 8, 9, 7].

In [1], the zeros for various partial sums were computed. The figure on the right gives the first 3000 zeros of ζ_{211}. This picture leads to many interesting questions and suggests a phenomenon in need of an explanation. The main goal of this project is to investigate the zeros of partial sums $\zeta_N(s)$ for various values of N. At the end, we will try to answer the following questions. a) *Are zeros of* $\zeta_N(s)$ *dense to the line* $\mathrm{Re}(s) = 0$*?* b) *Why are zeros concentrated on rays to the left of the line* $\mathrm{Re}(s) = 0$*?* c) *Does this phenomenon continue if we take* N *sufficiently large?* Partial sums of the Riemann zeta function are special cases of Dirichlet polynomials. This project allows students to explore both the theoretical aspects and the computational aspects of zeros of general Dirichlet polynomials.



REU Students' role and start-up topics. Basic calculus or advance high school math courses are enough to understand the background of this project. Beginner knowledge of Python or SageMath is a bonus. We will spend first two weeks on learning elementary number theory, basic complex analysis, and SageMath/Python. After this, students will be introduced various algorithms to compute the zeros of Dirichlet polynomials. Students will start their research by writing code to compute the zeros of the partial sums. Students will also be introduced to the relation between Dirichlet series and Dirichlet polynomials, the so-called approximate functional equations. This will be an essential component for students to investigate the distribution of zeros and give a theoretical explanation for the pattern of zeros seen in the figure.

References

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