RELATIONS BETWEEN DISCRETE AND CONTINUOUS SPECTRA OF SCHRÖDINGER OPERATORS

Being discrete and being continuous are two opposite properties of a set in the plane. However, there are situations in which the fact that one part of the set is discrete implies that the other part is continuous. The set below

has two parts: the discrete part (to the left of the vertical arrow), and the continuous one (to the right of the arrow). In general, one part is not related to the other. That is no longer true if this picture represents the spectrum of a Schrödinger operator!

There is a relation between the two parts of the spectrum. It is particularly simple if the potential $V(x)$ in the Schrödinger equation is negative. In this case, if the left part of the spectrum is discrete, then the right part is continuous. Moreover, the continuous part coincides with the half-line $[0, \infty)$. In the general case, one has to consider two Schrödinger operators, one of which is obtained from the other by flipping the sign of the electric potential $V(x)$ at every point x.

Theorem 1. *If the negative spectra of the two Schrödinger operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ are discrete, then both spectra contain every point of the interval $[0, \infty)$.*

(This theorem has some mathematical assumptions of the form $V \in L^p_{loc}(\mathbb{R}^d)$ that allow usual singularities of V appearing in physics.)

The rate of accumulation of eigenvalues to zero determines certain properties of the positive spectrum. If the negative eigenvalues tend to zero sufficiently fast, we can talk about absolute continuity of the positive part. Absolute continuity is a mathematical notion that is not easy to describe. An absolutely continuous spectrum can be seen in a rainbow in which one color is consecutively followed by another. The colors change from red to violet so gradually and smoothly, that one gets an impression that this passage is ”absolutely continuous”.

Theorem 2. *Let $d \geq 2$. Let V be a bounded function. Assume that the negative spectra of both operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ consist of isolated eigenvalues $\{\lambda_j^+\}_{j=1}^\infty$ and $\{\lambda_j^-\}_{j=1}^\infty$ satisfying the condition*

$$\sum_j |\lambda_j^+|^{1/2} + \sum_j |\lambda_j^-|^{1/2} < \infty.$$

Then the absolutely continuous spectrum of each of the two operators fills the positive half-line $[0, \infty)$.

The last line of the theorem should be understood in the sense that the density of the spectrum is positive almost everywhere on $[0,\infty)$. By the density of the spectrum we mean the derivative of the spectral measure μ (of the maximal type).

One of the reasons two operators are needed in such theorems is that the continuous spectrum of a Schrödinger operator with a decaying potential has only one edge. There are other interesting operators whose continuous spectra have two edges. For instance, one can consider the so-called discrete Schrödinger operator whose spectrum might (but does not have to) look like the set in the picture below.

In this case, the role of the solid half-line is played by the line segment whose edges are $-2d$ and $2d$ (here, d is the dimension of the space).

Theorem 3. *Let $d \geq 3$. If the spectrum of the discrete Schrödinger operator H outside of the interval $[-2d, 2d]$ is bounded and discrete, then the interval $[-2d, 2d]$ is contained in the spectrum of H.*