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Optimal Advertising and Pricing in a New-Product Adoption Model

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Abstract A model of new-product adoption is proposed that incorporates price and advertising effects. An optimal control problem that uses the model as its dynamics is solved explicitly to obtain the optimal price and advertising effort over time. The model has a great potential to be used in obtaining solutions and insights in a variety of differential game settings.

Keywords Durable goods sales · Advertising · Pricing · Optimal control · Dynamic programming

1 Introduction

In 1983, Sethi proposed a stochastic advertising model [1], which has found numerous applications such as [2–10], and has come to be known as the Sethi model. The deterministic version of that model is

$$\dot{x}(t) = \rho u(t)\sqrt{(1 - x(t))} - kx(t), \quad x(0) = x_0 \in [0, 1] \tag{1}$$

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where $x(t) \in [0, 1]$ is the fractional market potential (i.e., the rate of sales expressed as a fraction of the market potential or saturation level) and $u(t)$ is the rate of advertising effort (i.e., a control variable) at time t, and ρ and k are positive constants. Equation (1) states that changes in market share depend on two effects: response to advertising that acts positively (via the response constant ρ) on the unsold portion of the market, loss due to forgetting that acts negatively (via the decay constant k) on the sold portion of the market. It is the positive effect that was modified from $\rho u(1-x)$ in the Vidale-Wolfe model [11] to a larger non-linear effect $\rho u\sqrt{(1-x)}$. The difference $\rho u[\sqrt{(1-x)} - (1-x)]$ can be explained in part by an additional process of word-of-mouth communication between the individuals comprising the sold portion and those comprising the unsold portion. This effect is proportional to $ux(1-x)$. In our model, since the different term above is proportional to $ux(1-x)$ for small values of x, it could be considered to approximate the additional effect.

Furthermore, in [1] an optimal control problem is formulated with the dynamics (1) and the discounted profit functional

$$E\int_0^\infty [\pi x(t) - u^2(t)]\exp[-rt]dt, \tag{2}$$

where the revenue rate π and the discount rate r are given positive constants. The cost of advertising effort is a quadratic function; note that it is easy to scale the objective function so that the coefficient of u^2 is 1. The cost of advertising effort is essentially a production function for producing x. Assuming this to be a convex function implies increasing marginal costs in converting u into x. The maximization of profit was performed over the class of measurable controls $u(t),\ t \in [0, \infty)$.

In this paper, we present a model of cumulative sales that can be used to study new-product adoption of durable goods such as refrigerators and microwave ovens. New product adoption models of durable goods have been applied widely for demand forecasting and policy formulation by economists and marketers. A review of the literature is provided by Mahajan et al. [4]. Here, one typically assumes a fixed market potential of adopters. Once the product is purchased, and because it is a durable good that does not require immediate repurchase, the adopters exit the market. Thus, the untapped market shrinks over time. The goal of the marketing effort is to influence the remaining nonadopters. A well-known example of such a model is the Bass model of innovation diffusion [12]. On the other hand, with frequently purchased goods such as carbonated beverages and toothpaste, the adopters of the company's brand do not exit the market although they may switch to other brands. The 1983 Sethi model [1] gives an example of this.

In the next section, we define the notation and develop the model in terms of price charged and advertising effort made, introduce an objective function, and solve the resulting optimal control problem. In Sect. 3, we consider the case when the demand function is linear in price. In Sect. 4, we solve the case with an isoelastic demand function. In Sect. 5, we list the sensitivity results obtained in Sects. 3 and 4, and conclude the paper along with some suggestions for extensions.

2 New-Product Adoption Model

Let $X(t)$ denote the cumulative sales at time t. Let $P(t) \geq 0$ denote the price charged at time t and $D(P)$ denote the demand when the price is P. It is usual to assume that $D'(P) < 0$. We then formulate the model as

$$\dot{X}(t) = \rho U(t) D(P(t)) \sqrt{1 - X(t)} - kX(t), \quad X(0) = X_0 \in [0, 1], \tag{3}$$

where $\dot{X}(t)$ represents the rate of adoption or sales at time t, and it depends on $X(t)$, the advertising effort $U(t)$, and the price $P(t)$ at time t. In the case of durable goods, the decay term $kX(t)$ can be considered to approximate the effect of product breakage and returns. Note that the case when the total market is M can be reduced to (3) by defining $X(t)$ as the fraction of the cumulative market captured by time t and adjusting the parameter ρ accordingly.

Although the state variable $X(t)$ in (3) is different from $x(t)$ in (1), the structure of the differential equation (3) is similar to that of (1), except for the additional demand term $D(P(t))$. It is this structure that results in a very tractable model.

Let us now relate this model to the new-product adoption models that have appeared in the literature. In these models of durable goods, the decay term is usually assumed to be insignificant, and is ignored. Thus, in the remainder of this paper, we will also set $k = 0$ and consider the resulting model

$$\dot{X}(t) = \rho U(t) D(P(t)) \sqrt{1 - X(t)}, \quad X(0) = X_0 \in [0, 1]. \tag{4}$$

The Bass model [12] is given by

$$\dot{X} = a(1 - X) + bX(1 - X), \tag{5}$$

where a and b are positive constants. These coefficients may depend on pricing and advertising policies. A number of papers articulating this dependence have appeared in the literature. These are reviewed in Feichtinger et al. [13]. In particular, Robinson and Lakhani [14] set $a = \rho D(P)$ and $b = D(P)$; Horsky and Simon [15] set $a = a_1 + a_2 \log U$, where a_1 and a_2 are constants, and Sethi and Bass [16] set $a = D(P)$ and $b = 0$.

If we set $a = b = \rho U D(P)$ and approximate the term $(1 - X) + X(1 - X)$ by $\sqrt{1 - X}$, as explained in Sect. 1, then (5) reduces to (3).

In the next section, we assume the demand to be linear in price. We formulate the objective function and solve the resulting optimal control problem to obtain the optimal price and advertising policies.

3 Optimal Solution with Linear Demand

In this section, we set $D(P) = (1 - \eta P)$ in (3) with $\eta > 0$, which gives us the dynamics

$$\dot{X}(t) = \rho U(t)(1 - \eta P(t)) \sqrt{1 - X(t)}, \quad X(0) = X_0 \in [0, 1]. \tag{6}$$

Note that the *price elasticity of demand* at a given P is $-\eta P/(1-\eta P)$. This varies from perfectly inelastic at $P=0$ to perfectly elastic at $P=1/\eta$. Furthermore, the demand is inelastic in the range $P \in (0, 1/2\eta)$ unit elastic at $P=1/2\eta$, and elastic in the range $P \in (1/2\eta, 1/\eta)$. We should also mention that $D(P)=1-\eta P$ is the most general linear demand in our case, since $D(P)=\eta_1-\eta_2 P$ in (3) can be reduced to our case by replacing $\rho\eta_1$ by ρ and η_2/η_1 by η.

If we assume the cost of advertising U^2 results in an advertising effort of U, and for convenience in exposition, we also assume zero production cost, then the firm would like to maximize the present value of its profit, i.e.,

$$\max_{P(\cdot),U(\cdot)\geq 0} \int_0^\infty e^{-rt}[P(t)\dot{X}(t)-U^2(t)]dt \tag{7}$$

subject to (6).

It is important to contrast (7) with the objective function (2) used in Sethi [1]. Since the state variable $X(t)$ is cumulative sales, it is the current rate of sales $\dot{X}(t)$ that is used in (7), whereas in (2) the state variable $x(t)$ is used directly since it is the current rate of sales.

We now use dynamic programming to solve the optimal control problem given by (6) and (7). For this, let $V(X)$ denote the value function at any time $t \geq 0$ with $X(t)=X$. It satisfies the HJB equation (see e.g., Fleming and Rishel [17] or Sethi and Thompson [18])

$$rV=\max_{P,U\geq 0}[\rho UP(1-\eta P)\sqrt{1-X}-U^2+V_X\rho U(1-\eta P)\sqrt{1-X}],$$

which can be rewritten as

$$rV=\max_{P,U}[(P+V_X)\rho U(1-\eta P)\sqrt{1-X}-U^2]. \tag{8}$$

The first-order conditions (FOC) for optimal P and U can be solved to give

$$P=\frac{1-\eta V_X}{2\eta} \tag{9}$$

and

$$U=\frac{\rho(1+\eta V_X)^2\sqrt{1-X}}{8\eta}. \tag{10}$$

Substituting (9) and (10) in (8) gives

$$rV=\frac{\rho^2(1+\eta V_X)^4(1-X)}{64\eta^2}. \tag{11}$$

We will try a solution of the form

$$V(X)=\alpha(1-X), \tag{12}$$

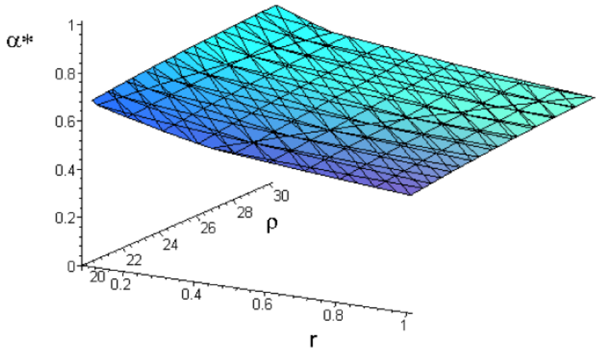

Fig. 1 α^* vs. r and ρ

where we need to determine α. It is obvious that we should have $\alpha > 0$. Substituting (12) and $V_x = -\alpha$ in (11), we get the quartic equation

$$(1-\eta\alpha)^4 = \frac{64r\eta^2}{\rho^2}\alpha \tag{13}$$

in α. Using Descarte's rule of signs, we can determine that there are 0, 2 or 4 positive roots for α. However, from (9) we know that $P = (1+\eta\alpha)/2\eta$ and, since $0 \le P \le 1/\eta$ and $\alpha > 0$, we are only interested in a value of α satisfying $0 \le \alpha \le 1/\eta$. Plotting graphs of $(1-\eta\alpha)^4$ and $(64r/\rho^2)\alpha$ against α, it is easy to see that there is only one positive root, say α^*, in the interval $[0, 1/\eta]$. Moreover, $\alpha^* = 0$ and $\alpha^* = 1/\eta$ do not satisfy (13), so we must have $0 < \alpha^* < 1/\eta$. With this α^*, the optimal price and advertising are

$$P^*(X) = \frac{1+\eta\alpha^*}{2\eta}, \qquad U^*(X) = \frac{\rho(1-\eta\alpha^*)^2\sqrt{1-X}}{8\eta}. \tag{14}$$

We see that the optimal price does not vary with X and is thus a constant and the advertising expenditure $U^{*2}(X)$ is proportional to the uncaptured market $(1-X)$.

For a given η, r, and ρ, it is very easy to solve (13) for the value of $\alpha^* \in (0,1)$. For example, if $\eta = 1$, $r = 0.15$, and $\rho = 15$, we obtain $\alpha^* = 0.6$. With this, $P^* = 0.8$ and $U^*(X) = 0.3\sqrt{1-X}$. In Fig. 1, we plot α^* against r and ρ.

Since we have an explicit solution in terms of α^* that can be easily computed, we are able to perform the sensitivity analysis with respect to the problem parameters. First, observe that the maximum profit associated with the total market is $V(0) = \alpha^*$. This is when we begin with $X(0) = X_0 = 0$ and follow the optimal price and advertising policies. Furthermore, the marginal value V_X of the cumulative sales is $-\alpha^* < 0$. This means that at any time t, if $X(t)$ changes from X to $X + \delta X$, we will have less remaining market to capture and, as a result, the total profit discounted to time t will decline by $\alpha^*\delta X$.

It is also easy to obtain $\partial\alpha^*/\partial r$, $\partial\alpha^*/\partial\rho$ and $\partial\alpha^*/\partial\eta$ by implicitly differentiating (13). We will illustrate this sensitivity with respect to r, and then simply give the results for the others since their derivations are similar. By differentiating (13) with respect to r, we get

$$\frac{\partial\alpha^*}{\partial r}\left[-4\eta(1-\eta\alpha^*)^3-\frac{64}{\rho^2}r\eta^2\right]=\frac{64}{\rho^2}\alpha^*\eta^2.$$

But from (13),

$$(1-\eta\alpha^*)^3=64r\eta^2\alpha^*/\rho^2(1-\eta\alpha^*).$$

Thus, substituting this in the above, we get

$$\frac{\partial\alpha^*}{\partial r}=-\frac{\alpha^*}{r}\left[\frac{1-\eta\alpha^*}{1+3\eta\alpha^*}\right]<0. \tag{15}$$

Similarly, we have

$$\frac{\partial\alpha^*}{\partial\rho}=\frac{2\alpha^*}{\rho}\left[\frac{1-\eta\alpha^*}{1+3\eta\alpha^*}\right]>0 \tag{16}$$

and

$$\frac{\partial\alpha^*}{\partial\eta}=-\frac{2\alpha^*}{\eta}\left[\frac{1+\eta\alpha^*}{1+3\eta\alpha^*}\right]<0. \tag{17}$$

From (15) and (16), we see that α^* decreases in r and increases in ρ. Thus, the optimal price P^* decreases as the discount rate r increases, and it increases as the advertising response constant ρ increases. The effect of the price sensitivity η on P^* is also easy to establish. We know from (17) that α^* decreases as η increases. Then, $P^*=(1/2)(1/\eta+\alpha^*)$ is easily seen to decrease as η increases.

As for the advertising policy, it is clear that the advertising effort $U^*(X)$ at any given X decreases as X increases. This is quite intuitive since advertising is less effective as more and more of the market potential is captured. Eventually and asymptotically the entire market is captured, i.e., $X(t)\to 1$ as $t\to\infty$. Also, for any fixed X, the advertising expenditure increases as the discount rate r increases. It is clear from (14) that $U^*(X)$ increases as the discount rate r increases. In order to establish how $U^*(X)$ changes with ρ, we need to take the derivative of the term $\rho(1-\eta\alpha^*)^2$ with respect to ρ. Taking this derivative and substituting for $\partial\alpha^*/\partial\rho$ from (16), we see, in view of the fact $\eta\alpha^*<1$, that

$$\frac{\partial}{\partial\rho}\left[\rho(1-\eta\alpha^*)^2\right]=\frac{1-\eta\alpha^*}{1+3\eta\alpha^*}(1-\eta\alpha^*+4\eta\alpha^{*2})>0. \tag{18}$$

Thus, $U^*(X)$ for any given X increases in the advertising response parameter ρ. This is an intuitively appealing result, but not so obvious because on one hand advertising effort is more effective and, on the other, the cost of that effort is quadratic. Finally, we take the derivative of the term $(1-\eta\alpha^*)^2/\eta$ with respect to η to see how $U^*(X)$ will change when η changes. After a lengthy calculation and substitution for $\partial\alpha^*/\partial\eta$ from (17), we can derive

$$\frac{\partial[(1-\eta\alpha^*)^2/\eta]}{\partial\eta}=-\frac{(1-\eta\alpha^*)^2(1+\eta\alpha^*)}{\eta^2(1+3\eta\alpha^*)}<0. \tag{19}$$

Thus, the advertising effort for any given X decreases as the demand becomes more sensitive to the price charged.

4 Optimal Solution with Isoelastic Demand

In this section, we consider an isoelastic demand $D(P) = P^{-\eta}$, where $-\eta$ represents the price elasticity of demand and we assume $\eta > 1$ so that the assumed demand is elastic.

The optimal control problem is (7) subject to

$$\dot{X}(t) = \rho U(t) P(t)^{-\eta} \sqrt{1 - X(t)}, \quad X(0) = X_0 \in [0, 1]. \tag{20}$$

The HJB equation (8) is now replaced by

$$rV - \max_{P,U \geq 0} \left[(P + V_X) \rho U P^{-\eta} \sqrt{1 - X} - U^2 \right]. \tag{21}$$

The FOC give

$$P = \frac{\eta V_X}{1 - \eta}, \qquad U = \frac{\rho \sqrt{1 - X}}{2\eta} \left(\frac{\eta V_X}{1 - \eta} \right)^{1-\eta}. \tag{22}$$

Once again, we conjecture a solution of form (12). Substitution of V, V_X, P, and U into (21) results in

$$\alpha^* = \left[\frac{\rho^2}{2r\eta^2} \left(\frac{\eta - 1}{\eta} \right)^{2\eta - 2} \right]^{\frac{1}{2\eta - 1}}. \tag{23}$$

With α^* obtained in (23), we can write the optimal price

$$P^* = \frac{\eta \alpha^*}{\eta - 1}, \tag{24}$$

which is a constant, and

$$U^*(X) = \frac{\rho}{2\eta} \left(\frac{\eta \alpha^*}{\eta - 1} \right)^{1-\eta} \sqrt{1 - X}. \tag{25}$$

To determine how these policies change with respect to the various parameters, first we obtain how α^* varies with these parameters. It is clear from (23) and (24) that, as in Sect. 3, the value α^* of the total market and the optimal price P^* decrease in r and increase in ρ.

As for the optimal advertising $U^*(X)$, it decreases as more and more market has been captured. For any given X, we can easily see from (25) that $U^*(X)$ increases as r increases. This is reasonable since the higher the discount rate, the lower is the value of future sales, and thus advertising effort increases to increase the sales early on to increase current earnings. For the effect of a change in ρ on $U^*(X)$, we need to obtain the derivative of $\rho\alpha^{*(1-\eta)}$ with respect to ρ. We can show that

$$\frac{\partial(\rho\alpha^{*(1-\eta)})}{\partial\rho} = \frac{1}{2\eta-1}\left[\frac{1}{\alpha^{*(\eta-1)}}\right] > 0,$$

which means that the advertising effort $U^*(X)$ increases as ρ increases.

The sensitivity with respect to the elasticity η is somewhat more complicated. We begin with the examination of how α^* changes with respect to η. Differentiating (23) with respect to η and simplifying, we can obtain

$$\frac{\partial\alpha^*}{\partial\eta} = \frac{2\alpha^*}{2\eta-1}\log\frac{\eta-1}{\eta\alpha^*} = -\frac{2\alpha^*}{2\eta-1}\log P^*. \tag{26}$$

This means that the value of the market decreases in η if $P^* > 1$ and increases in η if $P^* < 1$.

To determine how $U^*(X)$ changes with η, we need to take the derivative of the term $(\frac{1}{\eta})(\frac{\eta\alpha^*}{(1-\eta)})^{1-\eta}$ with respect to η. Taking the derivative, using (26), and simplifying, we can show that

$$\frac{\partial}{\partial\eta}\left[\frac{1}{\eta}\left(\frac{\eta\alpha^*}{\eta-1}\right)^{1-\eta}\right] = -\frac{P^{*(1-\eta)}}{\eta(2\eta-1)}\log P^*. \tag{27}$$

Thus, the advertising effect decreases in η so long as $P^* > 1$.

Whether α^* and $U^*(X)$ increase or decrease with respect to η depends on whether $P^* < 1$ or $P^* > 1$. But P^* itself depends on η. To examine how P^* changes with respect to η, we can show that

$$\frac{1}{P^*}\frac{\partial P^*}{\partial\eta} = -\frac{1}{\eta-1} - \frac{2}{2\eta-1}\log P^*.$$

Thus, $\partial P^*/\partial\eta \le 0$ if $P^* \ge e^{\frac{2\eta-1}{2\eta-2}}$ and $\partial P^*/\partial\eta > 0$ otherwise. We also conclude that $P^* \to \infty$ as $\eta \to 1$ and $P^* \to 1$ as $\eta \to \infty$. To get further insight into the behavior of P^* as η changes, we perform numerical analysis.

We set $\rho^2/2r = 2 > 1$ and plot P^* against η in Fig. 2. This reveals that as η increases from 1, optimal price P^* decreases from the value of infinity to below 1. At some value $\hat{\eta}(2)$, the price bottoms out, and for $\eta > \hat{\eta}$, it starts increasing monotonically and asymptotically to the value of 1. Thus, for $1 < \eta < \hat{\eta}(2)$, P^* decreases in η, and for $\eta > \hat{\eta}(2)$, P^* increases in η, where $\hat{\eta}(\rho^2/2r)$ is a constant that depends on the value of $\rho^2/2r$. Moreover, the value $\hat{\eta}(\rho^2/2r)$ can be obtained by solving the transcendental equation

$$\left(\frac{\eta}{\eta-1}\right)\left[\frac{\rho^2}{2r\eta^2}\left(\frac{\eta-1}{\eta}\right)^{2\eta-2}\right]^{\frac{1}{2\eta-1}} = e^{\frac{2\eta-1}{2\eta-2}}$$

in η. The second curve in Fig. 2 is obtained when we set $\rho^2/2r = 1/2 < 1$. It appears that the behavior of α^* versus ρ and r is similar for any value of $\rho^2/2r$.

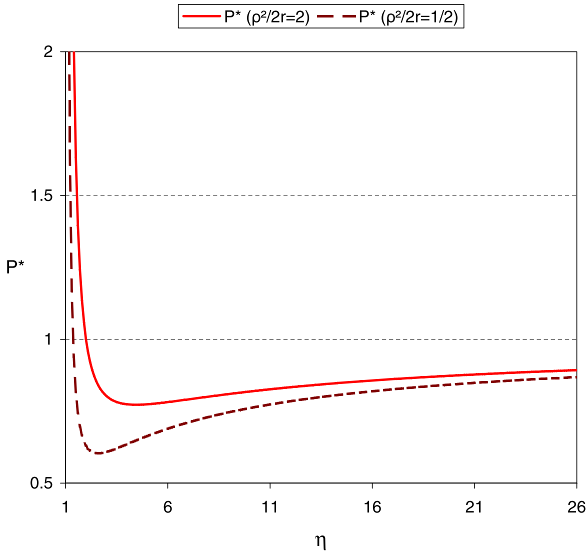

Fig. 2 P^* vs. η when $\rho^2/2r = 1/2$ and $\rho^2/2r = 2$

Table 1 Parameter sensitivity results

	Linear demand			Isoelastic demand		
	r	ρ	η	r	ρ	η
α^*	$-$	$+$	$-$	$-$	$+$	$-$ if $P^* > 1$ $+$ if $P^* < 1$
P^*	$-$	$+$	$-$	$-$	$+$	$-$ if $\eta < \hat{\eta}$ $+$ if $\eta > \hat{\eta}$
$U^*(X)$	$+$	$+$	$-$	$+$	$+$	$-$ if $P^* > 1$ $+$ if $P^* < 1$

In Table 1, we summarize the results of the sensitivity analysis.

5 Extensions and Concluding Remarks

In this brief paper, we have developed a new-product adoption model in terms of price and advertising. We formulate two optimal control problems—one for linear demand and the other for isoelastic demand—that use the newly developed model as the dynamics of sales and advertising, and provide their solutions explicitly. We have obtained comparative statics that examine the effects of parameter changes on the value of the total market, the optimal price, and the optimal advertising effort. More specifically, we show that the value of the market and the optimal price decrease in the discount rate and increase in the advertising response constant. In the linear demand case, we show that the market value, the optimal price, and the optimal advertising decreases with the price sensitivity. In the isoelastic demand case, these results hold if the price elasticity demand is low. Otherwise, the opposite holds.

Because of the explicitness of the solution, it is expected that the new model, like the earlier Sethi model [1], will be used in a variety of differential game settings with the possibility of obtaining solutions in closed form and eliciting useful insights into the price and advertising behavior of the firms playing the games; see, e.g., [19].

References

1. Sethi, S.P.: Deterministic and stochastic optimization of a dynamic advertising model. Optimal Control Appl. Methods **4**(2), 179–184 (1983)
2. Prasad, A., Sethi, S.P.: Competitive advertising under uncertainty: stochastic differential game approach. J. Optim. Theory Appl. **123**(1), 163–185 (2004)
3. Bass, F.M., Krishnamoorthy, A., Prasad, A., Sethi, S.P.: Generic and brand advertising strategies in a dynamic duopoly. Mark. Sci. **24**(4), 556–568 (2005)
4. Mahajan, V., Muller, E., Bass, F.M.: New product diffusion models in marketing: A review and directions for research. J. Mark. Res. **54**(1), 1–26 (1990)
5. Naik, P.A., Prasad, A., Sethi, S.P.: Building brand awareness in dynamic oligopoly markets. Manag. Sci. **54**(1), 129–138 (2008)
6. Sorger, G.: Competitive dynamic advertising: a modification of the case game. J. Econ. Dyn. Control **13**, 55–80 (1989)
7. Chintagunta, P.K., Jain, D.: Dynamic duopoly models of advertising competition: estimation and a specification tests. J. Econ. Manag. Strategy **4**(1), 109–131 (1995)
8. Bass, F.M., Krishnamoorthy, A., Prasad, A., Sethi, S.P.: Advertising competition with market expansion for finite horizon firms. J. Ind. Manag. Optim. **1**(1), 1–19 (2005)
9. He, X., Prasad, A., Sethi, S.P., Gutierrez, G.J.: A survey of Stackelberg differential game models in supply and marketing channels. J. Syst. Sci. Syst. Eng. **16**(4), 385–413 (2007)
10. He, X., Prasad, A., Sethi, S.P.: Cooperative advertising and pricing in a stochastic supply chain: feedback Stackelberg strategies. Working Paper, University of Texas at Dallas (2007). Available at SSRN: http://ssrn.com/abstract=1069063
11. Vidale, M.L., Wolfe, H.B.: An operations research study of sales response to advertising. Oper. Res. **5**(3), 370–381 (1957)
12. Bass, F.M.: A new product growth model for consumer durables. Manag. Sci. **15**(5), 215–227 (1969)
13. Feichtinger, G., Hartl, R.F., Sethi, S.P.: Dynamic optimal control models in advertising: recent developments. Manag. Sci. **40**(2), 195–226 (1994)
14. Robinson, B., Lakhani, C.: Dynamic price models for new-product planning. Manag. Sci. **21**(10), 1113–1122 (1975)
15. Horsky, D., Simon, L.S.: Advertising and diffusion of new products. Mark. Sci. **2**(1), 1–17 (1983)
16. Sethi, S.P., Bass, F.M.: Optimal pricing in a hazard rate model of demand. Opt. Control Appl. Methods **24**, 183–196 (2003)
17. Fleming, W.H., Rishel, R.W.: Deterministic and Stochastic Optimal Control. Springer, New York (1975)
18. Sethi, S.P., Thompson, G.L.: Optimal Control Theory: Applications to Management Science and Economics, 2nd edn. Kluwer Academic, Boston (2000)
19. Prasad, A., Sethi, S.P.: Optimal pricing and advertising in a durablegood duopoly. Working Paper, University of Texas at Dallas (2008). Available at SSRN: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1114989