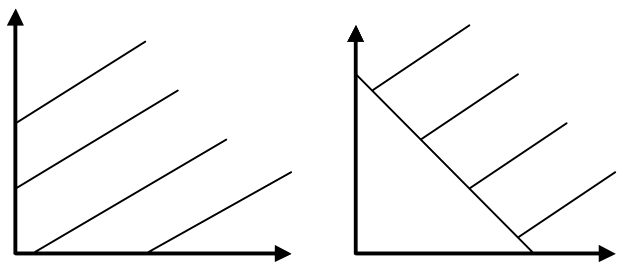
ECON 6202: Advanced Microeconomic Theory

Instructor: Dmitry Shapiro [1]

1 Utility and Preferences

1.1 Preferences

There are four elements of models of consumer choice. The first one is a choice set X,which is anything we could possibly like to consume. We will assume that 1)$X \neq \varnothing$; 2)$X \subseteq \mathbb{R}^n_+$ Examples of choice set X are given on the pictures below.



The second element of consumer choice is a feasible set B, which is a set of all consumption plans that are feasible given the circumstances (e.g. income and prices). We assume that $B \subset X$. The third element is a preference relation which reflects consumer's tastes for different objects of choices. The last one is "behavioral assumption", which means that consumer wants to find the best available alternative given his taste.

Consider the following, binary relation on X that we denote as " $\succcurlyeq$ ".Whenever we write $x \succcurlyeq y$ it means that "*x is at least as good as y" or "x is weakly preferred to y*". There are five axioms of the binary relation.

Axiom 1 (A1) *Completeness: $\forall x, y \in X$, either $x \succcurlyeq y$ or $y \succcurlyeq x$ (or both)*

Axiom 2 (A2) *Transitivity: $\forall x, y \in X$, $x \succcurlyeq y$ and $y \succcurlyeq z \Rightarrow x \succcurlyeq z$*

Definition 1.1 *Binary relation $\succcurlyeq$ that satisfies A1 and A2 is called preference relation.*

Based on $\succcurlyeq$ we can introduce two other binary relations:

- $x \succ y$ or "*x is strictly preferred to y*". We say that $x \succ y$ iff $x \succcurlyeq y$ and $y \not\succcurlyeq x$;
- $x \sim y$ or "*x is indifferent to y*". We say that $x \sim y$ iff $x \succcurlyeq y$ and $y \succcurlyeq x$.

Both $\succ$ and $\sim$ are *transitive* relations but not *complete*. For $\succ$ completeness requires that either $x \succ y$ or $y \succ x$, but if $x \sim y$, that doesn't hold. For $\sim$ completeness would mean that either $x \sim y$ or $y \sim x$, which is not true when $x \succ y$.

Let us introduce a couple of notations as follows:

- $\succcurlyeq (x_0) = \{x \mid x \in X, x \succcurlyeq x_0\}$ is called *at least as good* set;
- $\preccurlyeq (x_0) = \{x \mid x \in X, x \preccurlyeq x_0\}$ is called *no better than* set;

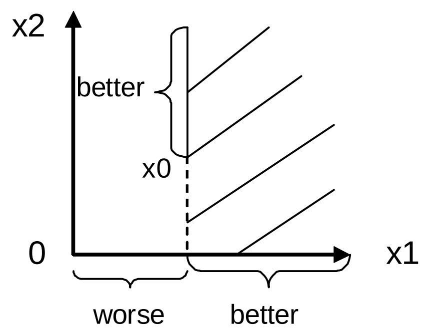
[1] These notes are preliminary, full of typos and in general are incorrect. Do not read or use!

- $\succ (x_0)$ is called *strictly better* set ;
- $\prec (x_0)$ is called *strictly worse* set;
- $\sim (x_0)$ is called *indifferent* set.

Axiom 3 (A3) *Continuity:* $\forall x \in \mathbb{R}^n_+$ *sets* $\succcurlyeq (x)$ *and* $x \preccurlyeq (x)$ *are closed sets in* $\mathbb{R}^n_+$.

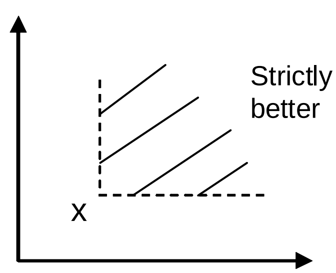
What is a closed set? See JR, Section A.1.3, p.417. When Axiom (A3) says that the set $\succcurlyeq (x)$ is closed it means the following. Take a sequence of bundles $y_1, \ldots, y_n, \ldots$. If $\forall n\ y_n \succcurlyeq x$ and $y_n \to y$ then $y \succcurlyeq x$.

Example 1.2 *An example of non-continuous preferences is lexicographic preference on* $X = \mathbb{R}^2_+$*:* $(x_1, x_2) \succ (y_1, y_2)$ *if either* $x_1 > y_1$ *or* $x_1 = y_1, x_2 > y_2$*. We define lexicographic preferences so that* $x \succcurlyeq y$ *if* $x_1 > y_1$ *or*



if $x_1 = y_1$ *and* $x_2 > y_2$*. The picture above shows that* $\succcurlyeq (x_0)$ *is not closed.*

Axiom 4 (A4) *Strict Monotonicity :* $\forall x, y \in \mathbb{R}^n_+$, *if* $x \geqslant y \Rightarrow x \succcurlyeq y$*; if* $x \gg y \Rightarrow x \succ y$.



Axiom 5 (A5) *Strict convexity: iff* $x \neq y$ *and* $x \succcurlyeq y$ *then* $tx + (1-t)y \succ y\ \forall t \in (0,1)$.

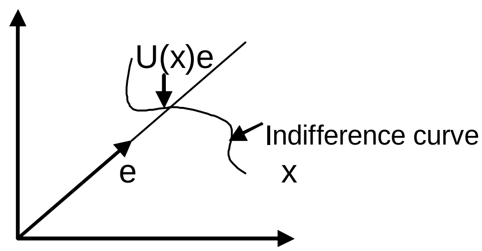
Essentially the strict convexity axiom states that "*the better set is convex*", which can be related to diminishing MRS.

1.2 Utility function

Definition 1.3 *A real-valued function* $u : \mathbb{R}^n_+ \to \mathbb{R}$ *is called a utility function representing* $\succcurlyeq$ *if* $\forall x, y \in \mathbb{R}^n_+ :\ u(x) \geqslant u(y) \Leftrightarrow x \succcurlyeq y$.

Theorem 1.4 *If preference relation $\succcurlyeq$ satisfies (A1)–(A4), then there exists a continuous $u : \mathbb{R}^n_+ \to \mathbb{R}$ which represents $\succcurlyeq$.*

Proof. We will explicitly construct the utility function that represents $\succcurlyeq$. Let $e = (1, 1...1)$ and define $u : \mathbb{R}^n_+ \to \mathbb{R}$, so that $u(x)e \sim x$.



This construction raises two questions: whether there exist such $u(x)$ and is it unique? The answer is yes to both. To prove the former, take t to be very small, then $te \prec x$(why?); take t to be very large, then $te \succ x$ (why?); then by continuity, $\exists t^*, t^*e \sim x$. As for the uniqueness, assume it does not hold. Then there exist two numbers t_1 and t_2 such that $t_1e \sim x$ and $t_2e \sim x$. Then $t_1e \sim t_2e$ which implies $t_1 = t_2$. To see the last application assume that $t_1 > t_2$. Then by strict monotonicity $t_1e > t_2e$, which is a contradiction.

The last thing to establish is that u represents $\succcurlyeq$. In order to do this we need to show: $u(x) \geqslant u(y) \Leftrightarrow x \succcurlyeq y$. which is true because

$$x \succcurlyeq y \;\Leftrightarrow\; u(x)e \sim x \succcurlyeq y \sim u(y)e \;\Leftrightarrow\; u(x)e \succcurlyeq u(y)e \;\Leftrightarrow\; u(x) \geq u(y)$$

This completes the proof of the theorem. ■

Question: Is the utility function that represents a particular $\succcurlyeq$ unique? The answer is no, $v(x) = u(x)+5$ would represent the same preferences as u. Furthermore, take any monotone increasing function $g : \mathbb{R} \to \mathbb{R}$. Then $g(u(x))$ is another utility function that represents $\succcurlyeq$. Indeed,

$$x \succcurlyeq y \;\Leftrightarrow\; u(x) \geq u(y) \;\Leftrightarrow\; g(u(x)) \geq g(u(y)).$$

1.3 Indifferent curves and MRS

Definition 1.5 *Indifferent curve is a set:* $\{x \in \mathbb{R}^+_n : u(x) = \bar{u}\}$

When $n = 2$ an indifferent curve is a set $IC = \{(x_1,x_2) : u(x_1,x_2) = \bar{u}\}$. Usually there are infinitely many IC: one for each $\bar{u}$.

Definition 1.6 *$MRS_{12}(x, y)$—marginal rate of substitution—is the rate at which the customer is willing to give up x_2 in exchange for x_1, so that he remains indifferent. More formally, $MRS_{12}(x, y)$ is the absolute value of the slope of the indifferent curve at (x, y).*

In the case of two goods as x_1 varies, x_2 varies as well in order to keep u constant. Thus we can define a function $x_2 = f(x_1)$ such that $u(x_1, f(x_1)) = \bar{u}$. By Definition above, $MRS_{12}(x_1, x_2)$ is the absolute value of function f'.

In what follows another representation of the MRS will be also useful. By Definition the MRS is

$$MRS_{12}(x_1,x_2) = \mid f'(x_1) \mid = -f'(x_1),$$

where function f is defined as

$$\bar{u} = u(x_{1,} x_2) = u(x_1, f(x_1)).$$

Taking its derivative with respect to x_1 we get

$$0 = \frac{\partial u(x_1, f(x_1))}{\partial x_1} + \frac{\partial u(x_1, f(x_1))}{\partial x_2} \cdot \frac{\partial f}{\partial x_1}$$

which implies

$$MRS_{12}(x_1, x_2) = -\frac{\partial f}{\partial x_1} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{MU_1}{MU_2}.$$

We've already seen that the same preferences can be represented by many different utility function. However, MRS does not depend on a particular choice of u. In other words the MRS of u and MRS of $g(u)$ are the same, where g is an increasing monotone function. Indeed, let

$$\hat{u}(x, y) = g(u(x, y)),$$

where $g : \mathbb{R}^2_+ \to \mathbb{R}$ and $g' > 0$. Then

$$\widehat{MRS_{12}}(x_1, x_2) = \frac{\partial \hat{u}/\partial x_1}{\partial \hat{u}/\partial x_2} = \frac{g'(u(x_1, x_2)) \cdot \partial u/\partial x_1}{g'(u(x_1, x_2)) \cdot \partial u/\partial x_2} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = MRS_{12}(x_1, x_2)$$

Another observation is that indifference curves do not depend on representation of u either. That is that

$$\{(x_1, x_2) : \; u(x_1, x_2) = \bar{u}\} = \{(x_1, x_2) : g(u(x_1, x_2)) = g(\bar{u})\}.$$

These two sets are the same, because g is monotone.

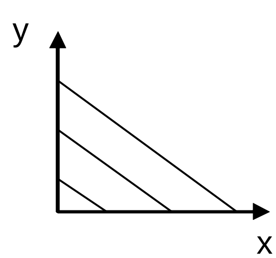
Example 1.7 *Cobb-Doublas utility function* $u(x, y) = x^\alpha y^\beta \quad \alpha > 0, \beta > 0$

$$MU_1 = \frac{\partial(x^\alpha y^\beta)}{\partial x} = y^\beta \cdot \frac{\partial x^\alpha}{\partial x} = y^\beta \alpha x^{\alpha-1};$$
$$MU_2 = \frac{\partial(x^\alpha y^\beta)}{\partial y} = \beta y^{\beta-1} x^\alpha;$$
$$MRS_{12}(x, y) = \frac{MU_1}{MU_2} = \frac{y^\beta \alpha x^{\alpha-1}}{\beta y^{\beta-1} x^\alpha} = \frac{\alpha y}{\beta x}.$$

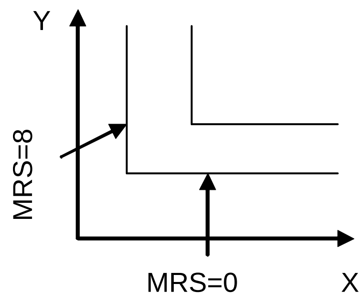
Example 1.8 *Perfect substitutes:* $u(x_1, x_2) = \alpha x_1 + \beta x_2$, *like Coke and Pepsi.*

$\bar{u} = u(x_1, x_2) = \alpha x_1 + \beta x_2 \Rightarrow x_2 = \frac{\bar{u}}{\beta} - \frac{\alpha}{\beta} \cdot x_1$

slope $= -\frac{\alpha}{\beta}$; $MRS = \frac{\alpha}{\beta}$; $MU_1/MU_2 = \frac{\alpha}{\beta}$



Example 1.9 *Perfect complements:* $u(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$, *like left shoe and right shoe. Another example,* x_1*—car,* x_2*—wheels so that* $u(x_1, x_2) = \min\{4x_1, x_2\}$. *For these preferences* $MRS = 0$ *or* ∞.



2 Consumer's problem

2.1 Marshallian Demand

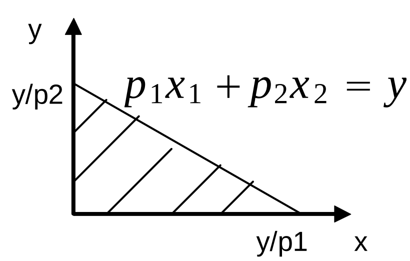
We define the consumer's problem as follows:

- *Consumption set* is $X = \mathbb{R}^n_+$; An element of the consumption set is an n-dimensional vector $x = (x_1, x_2...x_n)$, where x_i is the consumption of good i under x_i.
- *Budget set* B: for each of n goods there are prices $p = (p_1, p_2, ..., p_n)$ and $\forall i, p_i > 0$. Consumer's income is $y > 0$. Then budget set is the set of all consumpition plans that are affordable given price vector, p, and income, y. Formally, $B = \{x \in \mathbb{R}^n_+ \; p_1x_1 + \; p_2x_2... + \; p_nx_n \leqslant y\}$. For example, when $n = 2$

$$B = \begin{cases} p_1x_1 + p_2x_2 \leqslant y \\ x_1 \geqslant 0, x_2 \geqslant 0 \end{cases}$$

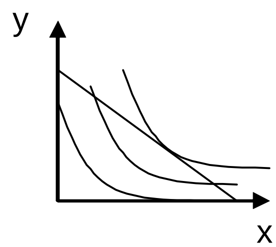
- *Consumer's problem* is to choose $x \in \mathbb{R}^n_+$ that maximizes the following problem:

$$\begin{cases} \max\limits_x u(x_1, x_2...x_n) \\ p_1x_1 + p_2x_2... + p_nx_n \leqslant y \\ x_i \geqslant 0 \; \forall i \end{cases} \tag{1}$$



B is non-empty, closed and bounded and, therefore, B is compact which means that the solution to the maximum problem exists. The maximization problem is equivalent to the problem of getting the bundle on the highest IC which satisfies budget constraints. Note: if $\succcurlyeq$ are monotone then budget constraint holds with equality.

Example 2.1 *Cobb-Douglas* $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ $0 < \alpha < 1$



$$\max L(x_1,x_2,\lambda) = x_1^{\alpha}x_2^{1-\alpha} + \lambda(y - p_1x_1 - p_2x_2)$$
$$\frac{\partial L}{\partial x_1} = \alpha x_1^{\alpha-1}x_2^{1-\alpha} + \lambda(-p_1) = 0$$
$$\frac{\partial L}{\partial x_2} = (1-\alpha)x_1^{\alpha}x_2^{-\alpha} + \lambda(-p_2) = 0$$
$$\frac{\partial L}{\partial \lambda} = y - p_1x_1 - p_2x_2 = 0$$
$$\Rightarrow x_1 = \frac{\alpha y}{p_1}, x_2 = \frac{y}{p_2}(1-\alpha)$$
$$\Rightarrow \lambda = \frac{1}{p_2}(1-\alpha)\left(\frac{\alpha y}{p_1}\right)^{\alpha}\left(\frac{y}{p_2}(1-\alpha)\right)^{-\alpha} = \frac{p_2^{\alpha-1}\alpha^{\alpha}}{(1-\alpha)^{\alpha-1}p_1^{\alpha}}$$

To see how this technique works for a more general utility functions consider the case of $n = 2$; we use Lagrange Multiplier and FOC to sloe this problem.

$$\begin{cases} \max u(x_1,x_2) \\ p_1x_1 + p_2x_2 \leqslant y \\ x_i \geqslant 0 \ \forall i = 1,2 \end{cases}$$

$$\max L(x_1,x_2,\lambda) = u(x_1,x_2) + \lambda(y - p_1x_1 - p_2x_2)$$
$$\frac{\partial L}{\partial x_1} = \frac{\partial u(x_1,x_2)}{\partial x_1} - \lambda p_1 = 0 \Rightarrow \frac{\partial u(x_1,x_2)}{\partial x_1} = \lambda p_1$$
$$\frac{\partial L}{\partial x_2} = \frac{\partial u(x_1,x_2)}{\partial x_2} - \lambda p_2 = 0 \Rightarrow \frac{\partial u(x_1,x_2)}{\partial x_2} = \lambda p_2$$
$$\frac{\partial L}{\partial \lambda} = y - p_1x_1 - p_2x_2 = 0$$
$$\Rightarrow MRS = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{p_1}{p_2}$$

In general case, we solve consumer's problem as follows:

$$\begin{cases} \max u(x_1,x_2...x_n) \\ p_1x_1 + p_2x_2... + p_nx_n \leqslant y \\ x_i \geqslant 0 \ \forall i \end{cases}$$

$$\max L(x_1, x_2...x_n, \lambda) = u(x_1, x_2..x_n) + \lambda(y - p_1x_1 - p_2x_2 - p_nx_n)$$

$$FOC: \frac{\partial L}{\partial x_1} = \tfrac{\partial u}{\partial x_1} - \lambda p_1 = 0 \Rightarrow \tfrac{\partial u}{\partial x_i} = \lambda p_i$$

$$\frac{\partial L}{\partial x_n} = \tfrac{\partial u}{\partial x_n} - \lambda p_n = 0 \Rightarrow \tfrac{\partial u}{\partial x_j} = \lambda p_j$$

$$\frac{\partial L}{\partial \lambda} = -\Sigma p_i x_i + y = 0$$

$$\Rightarrow MRS_{ij} = \frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \tfrac{p_i}{p_j}$$

However, Lagrange Multiplier doesn't work in some cases.

Example 2.2

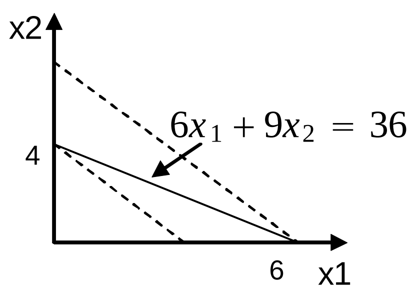
$$\max 3x_1 + 4x_2 \ st.6x_1 + 9x_2 = 36$$

$$L = 3x_1 + 4x_2 + \lambda(36 - 6x_1 - 9x_2)$$

$$\frac{\partial L}{\partial x_1} = 3 - 6\lambda = 0 \Rightarrow \lambda = \tfrac{1}{2}$$

$$\frac{\partial L}{\partial x_2} = 4 - 9\lambda = 0 \Rightarrow \lambda = \tfrac{4}{9}$$

It has no solution, so maximum will be achieved at the corner: (6,0).



Definition 2.3 *The solution to the consumer's maximization problem (1), $x(p, y)$, is called Marshllian demand function. It is a function that determines the optimal bundle given p and y.*

In example 2.1, $x_1(p_1, p_2, y) = \dfrac{\alpha y}{p_1}$; $x_2(p_1, p_2, y) = \dfrac{y}{p_2}(1 - \alpha)$. In particular, we can see that $x(kp, ky) = x(p, y)$. In fact this holds more generally. This is because the consumer's maximization problem given (p, y) is identical to the consumer's maximization problem given (kp, ky):

$$\begin{cases} \max u(x) \\ kp_1x_1 + kp_2x_2... + kp_nx_n \leqslant ky \end{cases} \Longleftrightarrow \begin{cases} \max u(x) \\ p_1x_1 + p_2x_2... + p_nx_n \leqslant y \end{cases}$$

Definition 2.4 $\mathbb{R}^n \to \mathbb{R}$ *is a homogeneous function of degree β if*

$$f(kx_1, kx_2...kx_n) = f(kx) = k^\beta f(x) = k^\beta f(x_1, x_2...x_n)$$

Since for the Marshallian demand function $x(kp, ky) = k^0 x(p, y) = x(p, y)$, it is homogenous of degree 0.

2.2 Indirect utility and expenditure functions

The utility function $u(x)$ is defined directly over the consumption set X, so it is usually called a direct utility function. However, we can ask the following question: what is the highest utility level that can be achieved given (p, y)? To answer this question we can construct a utility function which is a function of p and y: $v(p, y) = u(x(p, y))$. It is called indirect utility function, and it depends only on p and y, and not on the consumption bundle x.

Example 2.5 *From example 2.1 we know that for the Cobb-Douglas utility function the Marshallian demand is*

$$x(p, y) = \left(\frac{\alpha y}{p_1}, \frac{(1-\alpha)y}{p_2} \right).$$

Plugging it back into the utility function we get

$$u(x) = x_1^{\alpha} x_2^{1-\alpha} = \left(\frac{\alpha y}{p_1} \right)^{\alpha} \left(\frac{(1-\alpha)y}{p_2} \right)^{1-\alpha} = v(p, y)$$

Properties of $v(p, y)$:

1. v is homogeneous function of degree 0. This is because $v(kp, ky) = u(x(kp, ky)) = u(x(p, y)) = v(p, y)$.
2. $v(p, y)$ is increasing with respect to y.
3. $v(p, y)$ is decreasing with respect to p_i.
4. Roy's Identity:

$$x_i(p, y) = -\frac{\partial v(p, y)/\partial p_i}{\partial v(p, y)/\partial y};$$

Roy's Identity is very useful, for example, in evaluating the effects of price changes. From RI it follows that

$$\frac{\partial v}{\partial p_i} = -\frac{\partial v}{\partial y} \cdot x_i(p, y);$$
$$\frac{\partial v}{\partial p_j} = -\frac{\partial v}{\partial y} \cdot x_j(p, y).$$

Since we observe x_i and x_j we can evaluate whether it is better to subsidize industry i or industry j (for example in both industries prices are about to increase and the government has enough to money to subsidize only one industry from price increase).

To prove the RI, we will use envelope theorem:

Theorem 2.6 *(Envelope Theorem) Assume $x^*(a)$ solves $\max_x f(x, a)$, where a is a parameter like price. Define $g(a)$ as $g(a) = f(x^*(a), a)$, then*

$$\frac{dg(a)}{da} = \frac{\partial f(x^*(a), a)}{\partial a}$$

Proof. The proof is simple:

$$\frac{dg(a)}{da} = \frac{\partial f(x^*(a), a)}{\partial x} \cdot \frac{\partial x}{\partial a} + \frac{\partial f(x^*(a), a)}{\partial a} = \frac{\partial f(x^*(a), a)}{\partial a},$$

where the last equality holds because $x^*(a)$ is the solution to the maximization problem and so $\frac{\partial f(x^*(a), a)}{\partial x} = 0$. ■

Example 2.7 *Let $f(x,a) = ax - x^2$.*

$$\frac{\partial f(x,a)}{\partial x} = a - 2x = 0 \Rightarrow x^* = \frac{a}{2}$$
$$g(a) = f(x^*(a), a) = \frac{a^2}{2} - \frac{a^2}{4} = \frac{a^2}{2}$$
$$\frac{dg(a)}{da} = \frac{a}{2}$$

This was one way to find dg/da but it is long. Using the envelope theorem, we can immediately get $\frac{dg(a)}{da} = \frac{\partial f(x^*(a), a)}{\partial a} = x^*(a) = \frac{a}{2}$. That is, first, we take derivative of $f(x) = ax - x^2$ with respect to a, treating x as a constant so that $\frac{\partial f(x)}{\partial a} = x$,. Then we plug in optimal $x^*(a)$, which is equal to $\frac{a}{2}$.

Now use envelope theorem to prove RI. By definition

$$\begin{aligned}
v(p,y) &= \max_{x,\lambda}(u(x) + \lambda(y - p_1x_1 - ... - p_nx_n)) \\
\frac{\partial v(p,y)}{\partial p_i} &= -\lambda x_i; \qquad \frac{\partial v(p,y)}{\partial y} = \lambda \\
\Rightarrow x_i(p,y) &= -\frac{\partial v(p,y)/\partial p_i}{\partial v(p,y)/\partial y}
\end{aligned}$$

Another way of looking at consumer's maximization problem is to ask given p and $\bar{u}$, how much money is needed to achieve this level of $\bar{u}$? To find out we need to solve the problem

$$\begin{cases} \min_x p_1x_1 + p_2x_2... + p_nx_n \\ st.u(x) = \bar{u} \end{cases}$$

The Lagrangian and the First-Order Conditions are

$$\min_{x,\lambda} \Sigma p_i x_i + \lambda(\bar{u} - u(x_1, x_2...x_n))$$
$$\frac{\partial L}{\partial x_i} = p_i - \lambda\frac{u(x_1, x_2...x_n)}{\partial x_i} = 0$$
$$\frac{\partial L}{\partial \lambda} = \bar{u} - u(x_1, x_2...x_n) = 0$$

From the FOC it follows that

$$\frac{p_i}{p_j} = \frac{\partial u(x)/\partial x_i}{\partial u(x)/\partial x_j} = \frac{MU_i}{MU_j} = MRS_{ij}$$

Denote as $h(p,\bar{u})$ the solution to the above problem. It is called *Hicksian demand function.* $e(p,\bar{u}) = \Sigma p_i \cdot h(p,\bar{u})$ is cost of the bundle $h(p,\bar{u})$.

Example 2.8 *For the Cobb-Douglas utility function the minimization problem is*

$$\begin{cases} \min_x p_1x_1 + p_2x_2 \\ st.x_1^{\alpha}x_2^{1-\alpha} = \bar{u} \end{cases}$$

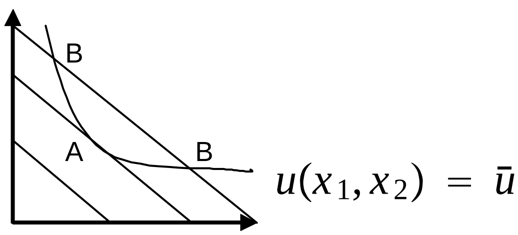
The FOC are

$$\frac{\partial L}{\partial x_1} = p_1 - \lambda\alpha x_1^{\alpha-1}x_2^{1-\alpha} = 0$$
$$\frac{\partial L}{\partial x_2} = p_2 - \lambda(1-\alpha)x_1^{\alpha}x_2^{-\alpha} = 0$$
$$\frac{\partial L}{\partial \lambda} = \bar{u} - x_1^{\alpha}x_2^{1-\alpha} = 0$$

Solving it we get

$$\begin{aligned}
h_1 &= \bar{u}\left(\frac{\alpha}{1-\alpha}\frac{p_2}{p_1}\right)^{1-\alpha}; \\
h_2 &= \bar{u}\left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \\
e(p,\bar{u}) &= p_1\bar{u}\left(\frac{\alpha}{1-\alpha}\frac{p_2}{p_1}\right)^{1-\alpha} + p_2\bar{u}\left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \\
&= \bar{u}p_1^{\alpha}p_2^{1-\alpha}\left(\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right)
\end{aligned}$$

Going back to the minimization problem. Intuitively, what we do is we fix IC and find the lowest budget line tangent to IC.



Properties of e and h functions:

1. $h(kp,\bar{u}) = h(p,\bar{u})$, homogeneous of degree 0 with respect to price.
2. $e(kp,\bar{u}) = ke(p,\bar{u})$, homogeneous of degree 1 with respect to price.
3. When $\bar{u}$ increases$\Rightarrow$ $e(p,\bar{u})$ increases too.
4. When p_i increases$\Rightarrow$ $e(p,\bar{u})$ increases too.
5. Shephard Lemma $\dfrac{\partial e(p,\bar{u})}{\partial p_i} = h_i(p,\bar{u}) > 0$. The proof is by envelope theorem

$$e(p,\bar{u}) = \min_{x,\lambda} \Sigma p_i x_i + \lambda(\bar{u} - u(x))$$
$$\frac{\partial e(p,\bar{u})}{\partial p_i} = h_i(p,\bar{u}) \text{ by envelope theorem}$$

6. $e(p,\bar{u})$ is concave with respect to the price vector.

Proof. We need to show that $e(\lambda p^1 + (1-\lambda)p^2, \bar{u}) \geqslant \lambda e(p^1,\bar{u}) + (1-\lambda)e(p^2,\bar{u})$. From the definition of e

$$\begin{aligned}
&e(p^*,\bar{u}) = p^* \cdot h^*(p^*,\bar{u}) \\
&= \lambda p^1 \cdot h^*(p^*,\bar{u}) + (1-\lambda)p^2 \cdot h^*(p^*,\bar{u}) \\
&\geqslant \lambda p^1 \cdot h^*(p^1,\bar{u}) + (1-\lambda)p^2 \cdot h^*(p^2,\bar{u}) \\
&= \lambda e(p^1,\bar{u}) + (1-\lambda)e(p^2,\bar{u})
\end{aligned}$$

■

When e is differentiable, $\dfrac{\partial^2 e(p,\bar{u})}{\partial p_i^2} \leqslant 0 \Rightarrow \dfrac{\partial}{\partial p_i}\left(\dfrac{\partial e(p,\bar{u})}{\partial p_i}\right) \leqslant 0 \Rightarrow \dfrac{\partial h_i(p,\bar{u})}{\partial p_i} \leqslant 0$, so Hicksian demand is *always* decreasing in its own price.

Given the definitions of x, h, v and e it is easy observe that the following equalities hold:

$$\begin{cases} v(p, e(p,\bar{u})) = \bar{u} \\ e(p, e(p,\bar{u})) = y \\ x(p, e(p,\bar{u})) = h(p,\bar{u}) \\ h(p, v(p,y)) = x(p,y) \end{cases} \tag{2}$$

The formal proof is given in JR p.40 and p.43. Intuitively, let us look at the first equality. Clearly, $v(p, e(p,\bar{u})) \geq \bar{u}$. Indeed by definition of e given p and $y(= e(p,\bar{u}))$ we can achieve at least $\bar{u}$. Since v is the maximum utility that can be achieved give p and y we have that $v(p,y) \geq \bar{u}$. Assume now that $v(p,y) > \bar{u}$. Then $v(p, y-\varepsilon) > \bar{u}$ by continuity of v. Consequently $e(p,\bar{u}) \leq y - \varepsilon$ which is a contradiction.

2.3 Compensating and Equivalent Variations.

Among the four functions that we consider, that is $x(p,y), v(p,y), h(p,u)$ and $e(p,u)$ the only one that is observed directly is Marshallian demand. From $x(p,u)$ we can derive the expenditure function which as a solution to the ordinary differential equation

$$\frac{de(p_1, \mathbf{p}, u^0)}{dp_1} = x_1(p_1, e(p_1, \mathbf{p}, u^0)).$$

Having an expenditure function we can analyze how consumer's well-being changes as a result of changes in price. Most importantly, the comparison can be made in terms of money.

Let's say the price vector changes from p^0 to p^1. The utility under old prices was u^0 and under new prices it is u^1. How does the price change affect the consumer? How much money should the government pay to the consumer to compensate him for price change? This amount is called *Compensating Variation (CV)* and is defined as

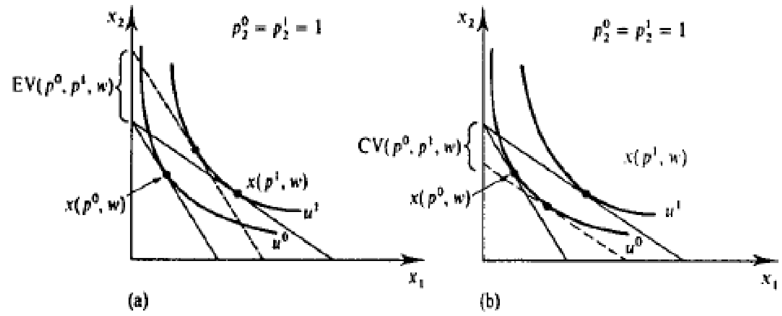
$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0) = \int_{p^1}^{p^0} h_i(p, p_{-i}, u^0)dp$$

Another way to measure the impact of price in monetary terms is to ask how much money would have to be taken away from the consumer *before* the price change to leave him as well off as he would be *after* the price change. This is called the *Equivalent Variation (EV)* in income since it is the income change that is equivalent to the price change in terms of the change in utility. Formally, it is defined as

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w = \int_{p^1}^{p^0} h_i(p, p_{-i}, u^1)dp$$

The CV and EV can be compensated graphically:

Consider a simple example. Assume that a consumer has a utility function $u(x_1, x_2) = \sqrt{x_1 x_2}$. We know the demand for a Cobb-Douglas function is $(\frac{y}{2p_1}, \frac{y}{2p_2})$. If original prices are $(1,1)$ and income is 100 then the optimal bundle then is $(50, 50)$. What should be a consumer's income, y_1 to keep the same utility as before when the prices change to $(2,1)$? Given the new prices consumer's demand is $(\frac{y_1}{4}, \frac{y_1}{2})$ and the new utility is $u_1 = \frac{y_1}{2\sqrt{2}}$. The old utility was 50. Therefore $y_1 = 100\sqrt{2} \approx 141$. Hence, the consumer would need about \$41 of additional money after the price change.



To calculate the EV we need to ask how much money would be necessary at $(1,1)$ to make the consumer as well of as he would be consuming $(25,50)$, or

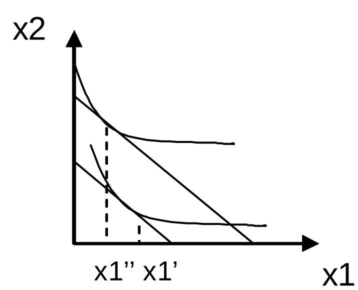
$$\left(\frac{y_1}{2}\right)^{1/2}\left(\frac{y_1}{2}\right)^{1/2} = 25^{1/2}50^{1/2}.$$

Solving for y_1 we get $y_1 = 50\sqrt{2} \approx 70$. Thus if the consumer had an income of \$70 at the original prices, he would be just as well of as he would be facing the new prices and having an income of \$100. The EV is $100 - 70 = 30$.

Note that CV and EV are not equal. That is, in general the amount of money that the consumer would be willing to pay to avoid a price change would be different from the amount of money that the consumer would have to be paid to compensate him for a price change. This is not surprising. After all, at different sets of prices a dollar is worth a different amount to a consumer since it will purchase different amount of consumption.

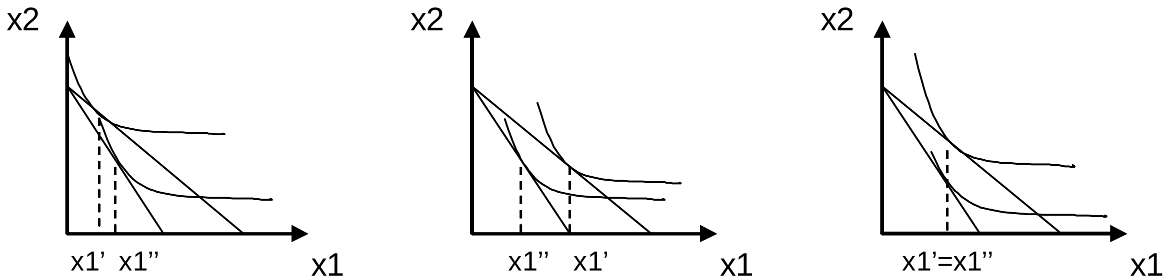
2.4 Properties of x. Substitution and Income Effect.

Definition 2.9 *If* $\dfrac{\partial x_i(p,y)}{\partial y} > 0$*, then* i *is called normal good; if* $\dfrac{\partial x_i(p,y)}{\partial y} < 0$*, then* i *is called inferior good.*

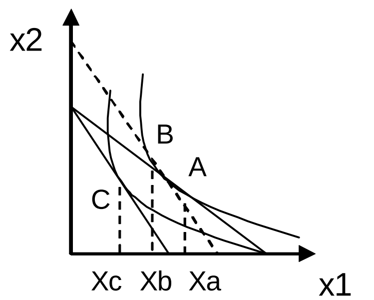


For example, x_1 could be a dinner in McDonalds, x_2 a dinner in fancy restaurant here.

As p_i increases, there are three possibilities for x_i: to increase, to decrease and to remain the same. All three possibilities are shown on the pictures below.



To see the effect of $\triangle p_i$ on x_i, decompose it into two effects: substitution effect (SE) and income effect(IE). On the picture below $x^B - x^A$ is SE, $x^C - x^B$ is income effect, and $x^C - x^A$ is total effect.



More formally, fix p and y and let $\bar{u} = v(p, y)$. As we've established above

$$h_i(p, \bar{u}) = x_i(p, e(p, \bar{u})).$$

Differentiating it with respect to p_i we get

$$\frac{\partial h_i(p, \bar{u})}{\partial p_i} = \frac{\partial x_i(p, e(p, \bar{u}))}{\partial p_i} + \frac{\partial x_i(p, e(p, \bar{u}))}{\partial y} \cdot \frac{\partial e(p, \bar{u})}{\partial p_i} \tag{3}$$

From Shephard Lemma and (2) we have that

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = h_i(p, \bar{u}) = h_i(p, v(p, y)) = x_i(p, y) \quad \text{and} \quad e(p, \bar{u}) = e(p, v(p, y)) = y$$

Thus we can re-write (3) as

$$\frac{\partial h_i(p, \bar{u})}{\partial p_i} = \frac{\partial x_i(p, y)}{\partial p_i} + \frac{\partial x_i(p, y)}{\partial y} \cdot x_i(p, y),$$

where each term represents corresponding total, substitution and income effects

$$\underbrace{\frac{\partial x_i(p, y)}{\partial p_i}}_{TE} = \underbrace{\frac{\partial h_i(p, \bar{u})}{\partial p_i}}_{SE} - \underbrace{\frac{\partial x_i(p, y)}{\partial y} \cdot x_i(p, y)}_{IE}.$$

This equality above is called **Slutsky Equation**. Using it we can analyze the effect of price on demand. First of all, substitution effect is always negative: $\frac{\partial h_i(p, \bar{u})}{\partial p_i} \leqslant 0$ (follows from concavity of e). As for the income effect, by definition 2.9, $IE < 0$ for normal goods and thus TE is always negative. For inferior good, however, $IE > 0$ which means that TE can be positive in which case the demand for the good increases as price increases (Giffin good). This logic shows that Giffin good is necessarily an inferior good.

For the derivation of Slutsky equation we took the equality $h_i(p, \bar{u}) = x_i(p, e(p, \bar{u}))$ and differentiated it with respect to p_i. We can also differentiate it with respect to p_j in which case we get

$$\frac{\partial x_i(p,y)}{\partial p_j} = \frac{\partial h_i(p,\bar{u})}{\partial p_j} - \frac{\partial x_i(p,y)}{\partial y} \cdot x_i(p,y),$$

and using the definitions below we can study the effect of change in p_j on x_i.

$$\begin{cases} \dfrac{\partial h_i(p,\bar{u})}{\partial p_j} > 0 \Rightarrow \ i \text{ and } j \text{ are Hicksian substitutes} \\ \dfrac{\partial h_i(p,\bar{u})}{\partial p_j} < 0 \Rightarrow i \text{ and } j \text{ are Hicksian complements} \\ \dfrac{\partial x_i(p,y)}{\partial p_j} > 0 \Rightarrow i \text{ and } j \text{ are gross substitutes} \\ \dfrac{\partial x_i(p,y)}{\partial p_j} < 0 \Rightarrow i \text{ and } j \text{ are gross complements} \end{cases}$$

For example if good i is normal good and goods i and j are Hicksian complements then the demand for good i always decreases when price for good j increases.

Remark 2.10 *Note that* $\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$ *(why?) while as you will show in the Problem Set it is not necessarily true for Marshallian demand, that is* $\frac{\partial x_i}{\partial p_j} \neq \frac{\partial x_j}{\partial p_i}$.

Empirically it is very convenient to study properties of goods and demand using elasticities.

Definition 2.11 *Elasticity* η_i *of demand for good* i *with respect to income is* % *change in quality demanded per* 1% *change in income, that is*

$$\eta_i = \frac{\triangle x_i / x_i}{\triangle y_i / y_i} = \frac{\triangle x_i}{\triangle y} \cdot \frac{y}{x}$$

In limit this expression becomes

$$\eta_i = \frac{\partial x_i(p,y)}{\partial y} \cdot \frac{y}{x_i(p,y)}.$$

Clearly, if $\frac{\partial x_i(p,y)}{\partial y} > 0$ then $\eta_i > 0$ and so i is a normal good. Similarly, if $\eta_i < 0$ then i is an inferior good. In a similar way we can define gross-price elasticity: $\varepsilon_{ij} = \frac{\partial x_i(p,y)}{\partial p_j} \cdot \frac{p_j}{x_i}$ and own-price elasticity: $\varepsilon_{ii} = \frac{\partial x_i(p,y)}{\partial p_i} \cdot \frac{p_i}{x_i}$. Then

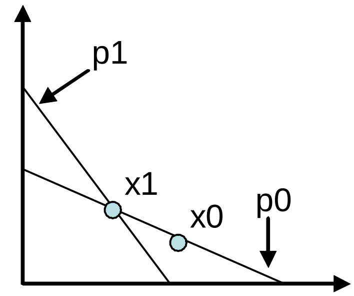
$$\begin{cases} \varepsilon_{ii} > 0 \Rightarrow \text{Giffin good} \\ -1 < \varepsilon_{ii} < 0 \Rightarrow \text{inelastic demand} \\ \varepsilon_{ii} \leqslant -1 \Rightarrow \text{elastic demand} \\ \varepsilon_{ij} > 0 \Rightarrow \text{gross substitutes} \\ \varepsilon_{ij} < 0 \Rightarrow \text{gross complements} \end{cases}$$

3 Revealed preferences

In Sections 1 and 2 we started with assumptions on preferences and from this we derived the observed properties of market demand (budget balance, price effects, etc.). In other words we began by assuming something things we cannot observe — preferences — to ultimately make predictions about something we can observe — consumer demand behavior.

An alternative way is to *start* and *finish* with observable behavior. It turns out that we can make some simple assumptions about *observable* choices made by consumers and from that it is possible to obtain a theory that is equivalent to the theory developed in Sections 1 and 2.

The idea is very simple. Assume that bundles x^0 and x^1 are affordable given p^0 and assume that x^0 is chosen. Now if given p^1 bundle x^1 is chosen then it *must* be the case that x^0 is no longer affordable.

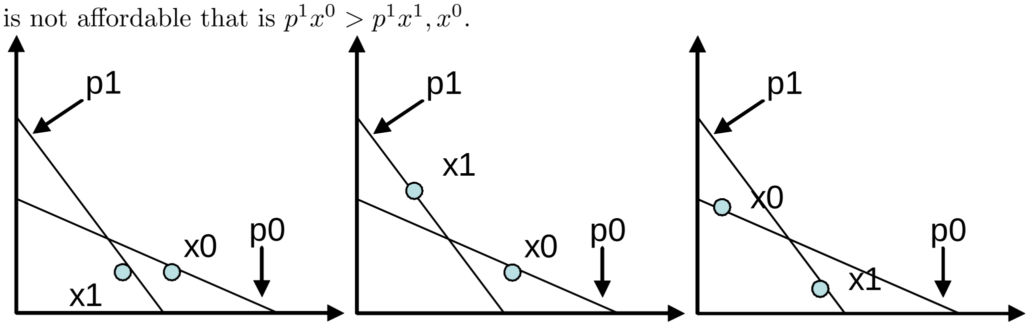


Definition 3.1 *If a consumer buys bundles x^0 instead of another affordable bundle x^1, then x^0 is revealed preferred to x^1.*

Definition 3.2 *Consumer's choice satisfies WARP (Weak Axiom of Revealed Preference) if for any pair of bundles x^0 and x^1 ($x^0 \neq x^1$), such that x^0 is chosen given p^0 and x^1 is chosen given p^1 it is the case that*

$$p^0x^1 \leqslant p^0x^0 \Rightarrow p^1x^0 > p^1x^1$$

In other words, if $p^0x^1 \leqslant p^0x^0$, then x^1 is affordable given p^0. If given p^1 bundle x^0 is not chosen then x^0 is not affordable that is $p^1x^0 > p^1x^1, x^0$.



The first and the second pictures satisfy WARP, the third one does not.

Denote as $x^c(p, y)$ the consumer's choice function given p and y. This is NOT demand function because we have not maximized utility. This is just a bundle chosen by the consumer given p and y. However, Marshallian demand is an example of a choice function.

Claim 3.3 *Marshallian demand $x^c(p, y)$ satisfies WARP.*

Proof. For simplicity assume that $\succcurlyeq$ are monotone and that optimal bundle is unique.

$$\left.\begin{array}{l} x^0 \ \max u \text{ given } p^0 \\ x^1 \ \max u \text{ given } p^1 \\ p^0x^1 \leqslant p^0x^0 \end{array}\right\} \Rightarrow u(x^0) > u(x^1)$$

Therefore, since x^1 is chosen given p^1, it must be that x^0 is not affordable. ■

More interesting questions is: if $x^c(p, y)$ satisfies WARP, can we find a utility function that would yield x^c as the outcome of utility maximization? If yes, then for all utility function that rationalizes the observed behavior $x^c(p, y)$. The answer is yes for $n = 2$, but it's not necessary for $n > 2$

Definition 3.4 *Strong Axiom of Revealed Preference(SARP): Consider $x^0, x^1...x^k$. Assume x^0 is revealed preferred to $x^1(p^0x^0 > p^0x^1)$; x^1 is revealed preferred to x^2, ..., x^{k-1} is revealed preferred to x^k. Then it cannot be the case that x^k is revealed preferred to $x^0(p^kx^k < p^kx^0)$.*

A useful feature of the SARP is that it rules out intransitive revealed preferences.

Theorem 3.5 *If x^c satisfies SARP, then we can rationalize $x^c(p, y)$ by some utility function.*

An immediate consequence of Theorem 3.5 is that a demand theory built on SARP (which puts restrictions on observable choice) is equivalent to the theory of demand based on utility maximization.

4 Choice under uncertainty

4.1 Lotteries

To describe uncertainty we assume that there is a set of outcomes $C = \{C_1, C_2...C_n\}$. For example, $C = \{$ nothing, trip to Italy, \$5000$\}$. Agent knows C and knows the probability of each outcome, but he doesn't know which outcome will occur.

Definition 4.1 *Lottery is a probability distribution over C. That is $L = (p_1, p_2...p_n)$, where $p_i = p(c_i)$, $\forall i$ $p_i \geqslant 0$ and $\Sigma p_i = 1$.*

For example, $L = (\underset{\text{nothing}}{\frac{1}{2}}, \underset{\text{Italy}}{\frac{1}{4}}, \underset{\$5000}{\frac{1}{4}})$.

Definition 4.2 *Let $\mathcal{L} = \{(p_1, p_2...p_n)\ \forall i\ p_i \geqslant 0,\ \Sigma p_i = 1\}$ denote the set of all lotteries. We will refer to these lotteries as simple lotteries because they directly assign probability to each outcome.*

Definition 4.3 *Compound lottery ia a lottery over lotteries. That is when with some probability you win one lottery and with another probability you win another lottery.*

Example 4.4 *Let $L_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $L_2 = (\frac{1}{2}, 0, \frac{1}{2})$ are two simple lotteries. Then lottery $L = \frac{1}{2}L_1 + \frac{1}{2}L_2$ is a compound lottery. Given L the probability of nothing $= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$; probability of Italy$= \frac{1}{4}$; probability of \$5000$= \frac{1}{4}$. Thus compound lottery L is equivalent to a simple lottery $(\underset{nothing}{\frac{1}{2}}, \underset{Italy}{\frac{1}{4}}, \underset{\$5000}{\frac{1}{4}})$.*

Consumer only cares about the final outcome, so we can look at simple lotteries. Indeed, if we have k simple lotteries: $L_1, L_2...L_k$ with $L_j = (p_1^j, p_2^j...p_n^j)$ then

$$\alpha_1 L_1 + \alpha_2 L_2 + ... + \alpha_k L_k = (\alpha_1 p_1^1 + \alpha_2 p_1^2 + ... + \alpha_k p_1^k, ...\alpha_1 p_n^1 + \alpha_2 p_n^2 + ... + \alpha_k p_n^k),$$

In other words any compound lottery is equivalent to a simple lottery.

4.2 Preferences over lotteries and expected utility

Example 4.5 *Consider a person who owns the house with value* \$500,000, *and there is* 3% *probability of hurricane to destroy the house. The person could choose between a lottery without house insurance (L_1) or with the insurance (L_2). Assume that the cost of the insurance is \$100,000 and that in the case of the hurricane the insurance fully covers the cost of the house. Then*

$$L_1 = \left\langle \begin{array}{l} 0(3\%) \\ \$500,000(97\%) \end{array} \right. \qquad L_2 = \left\langle \begin{array}{l} 500,000 - 100,000(3\%) \\ 500,000 - 100,000(97\%) \end{array} \right.$$

What lottery will the consumer choose? It depends on his preferences over the lotteries which we now are going to introduce.

Now we introduce $\succcurlyeq$ on $\mathcal{L}$. In fact, we can use our results from Section 1 where we proved that when $\succcurlyeq$ satisfy A1-A4 there exists u that represents $\succcurlyeq$. In the case of lotteries, the theorem becomes

Theorem 4.6 *If $\succcurlyeq$ on $\mathcal{L}$ satisfies A1–A3, then $\exists u : \mathcal{L} \to \mathbb{R}$ that represent $\succcurlyeq$, $L_1 \succcurlyeq L_2 \Leftrightarrow u(L_1) \succcurlyeq u(L_2)$.*

We do not need A4 (monotonicity axiom) because is not applicable to lotteries. If $L = (\frac{1}{2}, \frac{1}{2})$ is a lottery then $L' = (\frac{1}{2} + \varepsilon, \frac{1}{2} + \varepsilon)$ is not.

Our next goal is to show that the utility function over the lotteries can have very specific functional form. This is in contrast with normal bundles where utility can be essentially anything. To show that we will use an additional axiom.

Axiom 6 *Independence axiom (IA): $\succcurlyeq$ satisfies IA if $\forall L', L'', L$, $L \succcurlyeq L' \Leftrightarrow \alpha L + (1-\alpha)L'' \succcurlyeq \alpha L' + (1-\alpha)L''$.*

Note: IA *does not* hold for "real" goods. For example, assume there are three goods: left shoes, right shoes and dollars. Consider three bundles $x_1 = (0, 2, 0)$; $x_2 = (0, 0, 10)$ and $x_3 = (2, 0, 0)$. Then $x_2 \succcurlyeq x_1$. By IA it would be that $\frac{1}{2}x_2 + \frac{1}{2}x_3 = (1, 0, 5) \preccurlyeq \frac{1}{2}x_1 + \frac{1}{2}x_3 = (1, 1, 0)$, which seems somewhat strange.

Definition 4.7 *Utility function u: $\mathcal{L} \to \mathbb{R}$ has an expected utility form if there are n numbers $v_1, v_2, ..v_n$ such that $\forall L = (p_1, p_2, ..p_n)$*

$$u(L) = p_1 v_1 + p_2 v_2 + ... + p_n v_n.$$

A utility function with the expected utility form is called Neuman–Morgenstern utility function. If $L = C_i \Rightarrow u(L) = v_i$, so that v_i is the utility of outcome. The utility of a lottery is the expected value of utilities of outcomes.

Example 4.8 *Let $C = \{H, T\}, L = (\frac{1}{2}, \frac{1}{2}), u(H) = 1, u(T) = -1$. Then $u(L) = \frac{1}{2} \cdot 1 + \frac{1}{2}(-1) = 0$.*

If $u(\Sigma \alpha_j L_j) = \Sigma \alpha_j u(L_j)$, then it is called a linear function. We can show that vNM utility function is a linear function, that is if $L_1, ..L_k$ are simple lotteries and $L = \Sigma \alpha_i L_i = \alpha_1 L_1 + \alpha_2 L_2 + ... + \alpha_k L_k$, then

$$\begin{aligned} u(L) = & \underbrace{(\Sigma \alpha_j p_1^j)}_{\text{probability of } c_1} v_1 + \underbrace{(\Sigma \alpha_j p_2^j)}_{\text{probability of } c_2} v_2 + ... + (\Sigma \alpha_j p_n^j) v_n \\ & = \Sigma \alpha_j \underbrace{(p_1^j v_1 + p_2^j v_2 + ... + p_n^j v_n)}_{u(L_j)} = \Sigma \alpha_j u(L_j) \end{aligned}$$

Theorem 4.9 *If $\succcurlyeq$ on $\mathcal{L}$ satisfies A1–A3 and IA, then they admit utility representation of the EU form. That is: $\exists v_1, ...v_n$ such that $\forall L, L'$:*

$$\underset{(p_1, p_2, ..p_n)}{L} \succcurlyeq \underset{(p'_1, p'_2, ..p'_n)}{L'} \Leftrightarrow \Sigma p_i v_i \succcurlyeq \Sigma p'_i v_i.$$

Proof. Let $C = \{C_1, ...C_n\}$, and without loss of generality assume that $C_1 \succcurlyeq C_2 \succcurlyeq C_3 \succcurlyeq ... \succcurlyeq C_n$. Also note that, for example, C_1 is equivalent to the lottery $(1, 0, ...0)$.

Step 0: $\forall L$ $C_1 \succcurlyeq L \succcurlyeq C_n$ so that C_1 can be considered as the best and C_n as the worst lotteries. Denote them as $C_1 = L_B$ $\;C_n = L_W$.

Step 1: If $L \succcurlyeq L'$ and $\alpha \in [0, 1]$ then $L \succcurlyeq \alpha L + (1 - \alpha)L' \succcurlyeq L'$. This is by IA since $L = \alpha L + (1 - \alpha)L$.

Step 2: $\alpha, \beta \in [0, 1]$ $\alpha L_B + (1 - \alpha)L_W \succcurlyeq \beta L_B + (1 - \beta)L_W \Leftrightarrow \alpha \succcurlyeq \beta$.

Indeed,

$$\begin{aligned}
\alpha L_B + (1-\alpha)L_W &= \underbrace{(\frac{\alpha-\beta}{1-\beta})}_{\delta}L_B + \underbrace{(1-\frac{\alpha-\beta}{1-\beta})}_{1-\delta}(\beta L_B + (1-\beta)L_W) \\
&= \delta L_B + (1-\delta)(\beta L_B + (1-\beta)L_W) \\
&\succcurlyeq \delta(\beta L_B + (1-\beta)L_W) + (1-\delta)(\beta L_B + (1-\beta)L_W) \\
&= \beta L_B + (1-\beta)L_W
\end{aligned}$$

Where the first inequality follows from $L_B \succcurlyeq \beta L_B + (1 - \beta)L_W$.

Step 3: $\forall L$ $\exists \alpha_L \in [0, 1]$ such that $L \sim \alpha_L L_B + (1 - \alpha_L)L_W$.

The existence follows from continuity and uniqueness follows from step 2. Let $\{\succcurlyeq L\}$ is the set of all α's between 0 and 1 such that $L \succcurlyeq \alpha L_B + (1-\alpha)L_W$. By continuity this set is closed. Similarly $\{\preccurlyeq L\}$ is closed as well. At the same time by completeness $\{\succcurlyeq L\} \cup \{\preccurlyeq L\} = [0, 1]$. Thus there exists a common point of both sets which is the lottery that is equivalent to L. The uniqueness follows from step 2. From step 2 in particularly follows that if $\beta > \alpha$ then $\alpha L_B + (1 - \alpha)L_W \prec \beta L_B + (1 - \beta)L_W$

Step 4: Let $u(L) \overset{\text{def}}{=} \alpha_L$. Then $u(L)$ represents $\succcurlyeq$

$$\begin{gathered}
u(L) \succcurlyeq u(L') \Leftrightarrow \alpha_L \succcurlyeq \alpha'_L \\
\Leftrightarrow \alpha_L L_B + (1-\alpha_L)L_W \succcurlyeq \alpha'_L L_B + (1-\alpha'_L)L_W \Leftrightarrow L \succcurlyeq L'
\end{gathered}$$

Step 5: $u(L)$ is linear. $u(\beta L + (1 - \beta)L') = \beta u(L) + (1 - \beta)u(L')$

Indeed, we know that $L \sim u(L)L_B + (1 - u(L))L_W, L' \sim u(L')L_B + (1 - u(L'))L_W$ then,

$$\begin{aligned}
\beta L + (1-\beta)L' &= \beta[u(L)L_B + (1-u(L))L_W] + (1-\beta)[u(L')L_B + (1-u(L'))L_W] \\
&= [\beta u(L) + (1-\beta)u(L')]L_B + [\beta(1-u(L)) + (1-\beta)(1-u(L'))]L_W \\
&= [\beta u(L) + (1-\beta)u(L')]L_B + [1-\beta u(L) - (1-\beta)u(L')]L_W \\
&\Rightarrow u(\beta L + (1-\beta)L') = \beta u(L) + (1-\beta)u(L')
\end{aligned}$$

■

Example 4.10 $C = (C_1, C_2, C_3)$ $L = (p_1, p_2, p_3)$ $v_1 = v_2 = 1, v_3 = 0$, *then* $u(L) = p_1 + p_2$ *represents* $\succcurlyeq$ *and EU.* $\widetilde{u}(L) = u^2(L) = (p_1 + p_2)^2$ *still represents* $\succcurlyeq$*, but it is not in EU form.*

4.3 Money and risk aversion

In this section $C = \mathbb{R}_+$ $\;u(x) : \mathbb{R}_+ \to \mathbb{R}$, and we assume that $u(x)$ increases. A lottery $\widetilde{x}$ is a random variable with outcomes $x_1, ..x_n$ and probabilities $p_1, ...p_n$. Its mean payoff is $E(\widetilde{x}) = \Sigma p_i x_i$. The expected utility of $\widetilde{x}$ is

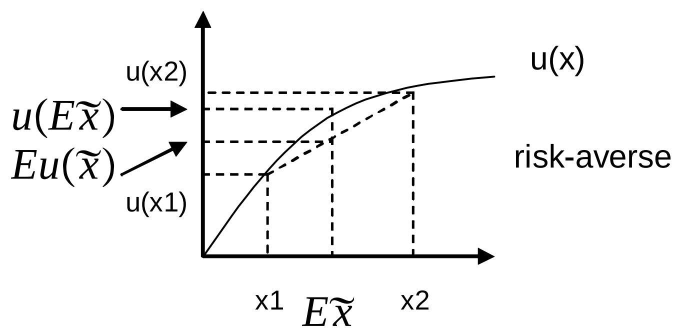
$$U(\widetilde{x}) = \Sigma p_i u(x_i) = Eu(\widetilde{x})$$

$$\widetilde{x}_1 \succcurlyeq \widetilde{x}_2 \Leftrightarrow Eu(\widetilde{x_1}) \geqslant Eu(\widetilde{x}_2)$$

We are interested in whether an individual prefers lottery $\widetilde{x}$ to certain payoff $E(\widetilde{x})$, that is when $\widetilde{x} \succcurlyeq E(\widetilde{x})$. Using utility representation, this is equivalent to

$$U(\widetilde{x}) = Eu(\widetilde{x}) \geqslant u(E(\widetilde{x})) = U(E(\widetilde{x}))$$

Definition 4.11 *An individual is strictly risk-averse if $\forall \widetilde{x}$ (with at least two outcomes) $E(u(\widetilde{x})) < u(E(\widetilde{x}))$;risk-loving if $E(u(\widetilde{x})) > u(E(\widetilde{x}))$; risk-neutral if $E(u(\widetilde{x})) = u(E(\widetilde{x}))$.*



Claim 4.12 *An individual is strictly risk-averse if and only if his utility function is strictly concave.*

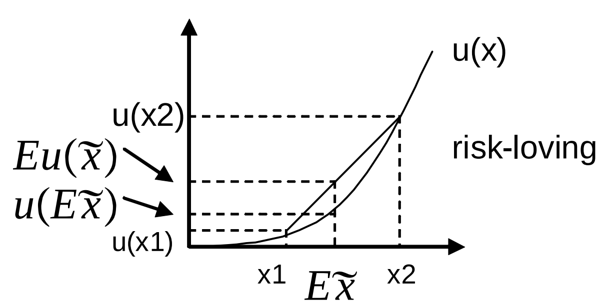
Sketch: a lottery with two outcomes x_1, x_2. By definition of concavity,

$$u(E\widetilde{x}) = u(p_1 x_1 + (1 - p_1)x_2) > p_1 u(x_1) + (1 - p_1)u(x_2) = Eu(\widetilde{x})$$

We can show it for n outcomes: when $u(\cdot)$ is strictly concave, $u(\Sigma p_i x_i) > \Sigma p_i u(x_i)$ from Jensen's inequality.

As we know, $u(\cdot)$ is strictly concave if $u''(\cdot) < 0$.

Theorem 4.13 *An individual is strictly risk-loving if and only if $u(\cdot)$ is strictly convex, that is $u''(\cdot) > 0$.*



Theorem 4.14 *An individual is risk-neutral if and only if $u(x) = a + bx$*

Example 4.15 *When $u(x) = x$, then*

$$\left.\begin{array}{l} E(\widetilde{x}) = \Sigma p_i x_i \\ u(E\widetilde{x}) = \Sigma p_i x_i \\ Eu(\widetilde{x}) = \Sigma p_i u(x_i) = \Sigma p_i x_i \end{array}\right\} \Rightarrow u(E\widetilde{x}) = Eu(\widetilde{x}) \text{ risk-neutral}$$

4.4 Applications

4.4.1 Insurance

Claim 4.16 *A RA individual will buy a full insurance for the fair price.*

Proof. Let w be initial wealth, π be the probability of loss, L be the lottery. If an individual receives \$1 for \$$\pi$ paid to insurance company we call it a fair price. This is because given this price the company's expected profit is 0: $\alpha\pi \cdot 1 - \alpha\pi = 0$

$$\underbrace{\left\langle \begin{array}{l} w - L(\text{with } \pi \text{ probability}) \\ w(\text{with } 1-\pi \text{ probability}) \end{array}\right.}_{\text{the lottery without insurance}} \quad \underbrace{\left\langle \begin{array}{l} w - L - \alpha\pi + \alpha \cdot 1(\text{with } \pi \text{ probability}) \\ w - \alpha\pi(\text{with } 1-\pi \text{ probability}) \end{array}\right.}_{\text{the lottery with insurance}}$$

$$Eu(\widetilde{x}) = (1-\pi)u(w - \alpha\pi) + \pi u(w - L + \alpha(1-\pi)) \to \max_{\alpha}$$
$$FOC: (1-\pi)(-\pi)u'(w-\alpha\pi) + \pi(1-\pi)u'(w - L + \alpha(1-\pi)) = 0$$
$$u'(w - L + \alpha(1-\pi)) - u'(w - \alpha\pi) = 0$$
$$u''(\cdot) < 0 \Rightarrow u'(\cdot) \text{ is strictly decreasing}$$
$$w - L + \alpha(1-\pi) = w - \alpha\pi \Rightarrow \alpha = L$$

Thus, the RA agent ends up with full insurance:

$$\left\langle \begin{array}{l} w - L - L\pi = w - L\pi(\text{with } \pi \text{ probability}) \\ w - L\pi(\text{with } 1-\pi \text{ probability}) \end{array}\right.$$

That is his payoff is the same in both states. ■

4.4.2 Stock market

Claim 4.17 *RA agent will always buy a risky asset with strictly positive excess return.*

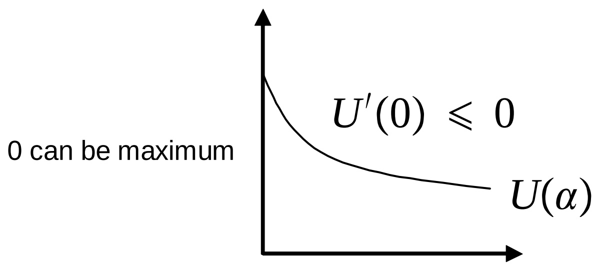
Proof. Let w be an initial wealth and assume that there are two assets: a riskless asset with return τ, that is invest \$1 receive \$$\tau$, and a risky asset with return $\widetilde{x}$. The question is how much to invest into the risky asset? Investing α units of risky assets gives income

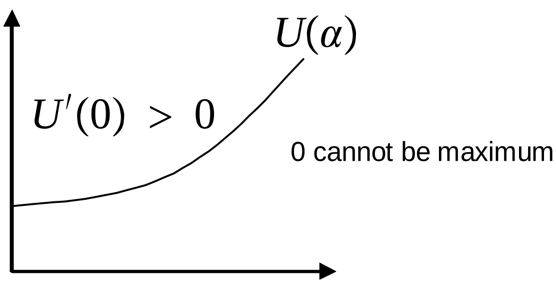
$$y = \alpha\widetilde{x} + \tau(w - \alpha) = \alpha(\widetilde{x} - \tau) + \tau w$$
$$Eu(\widetilde{x}) = \Sigma p_i u(\alpha(\widetilde{x} - \tau) + \tau w) \to \max_{\alpha}$$

First, if $\forall j\ x_j > \tau$, if on the other hand $\alpha^* = w$ and $\forall j\ x_j < \tau$ then $\alpha^* = 0$. Taking the first-order conditions we get

$$FOC: U'(\alpha) = \Sigma p_i(x_i - \tau)u'(\alpha(\widetilde{x} - \tau) + \tau w) = 0.$$

As we know these conditions are applicable only if we the solution is interior. When can we have a corner solution, specifically when $\alpha = 0$ is optimal? This would happen if $U'(0) \leqslant 0$ and $U''(0) \leqslant 0$. $U''(\alpha) = \underbrace{\Sigma p_i(x_i - \tau)^2}_{>0}\underbrace{u''(\alpha(\widetilde{x} - \tau) + \tau w)}_{<0} < 0$, so $\alpha = 0$ is optimal$\Leftrightarrow U'(0) \leqslant 0$.





$$
\begin{aligned}
U'(0) = \Sigma p_i(x_i - \tau)u'(\tau w) &\Leftrightarrow \Sigma p_i x_i = E(\widetilde{x}) \leqslant \tau \\
= \underbrace{u'(\tau w)}_{>0}\underbrace{(\Sigma p_i x_i - \tau \Sigma p_i)}_{\leqslant 0}
\end{aligned}
$$

When $E(\widetilde{x}) \leqslant \tau$, a RA individual doesn't buy a risky asset. When $E(\widetilde{x}) > \tau$, a RA individual will choose $\alpha > 0$. $E(\widetilde{x}) - \tau$ is called excess return because it shows how much the risky asset is more profitable then the riskless asset. ■

4.4.3 Risk-aversion coefficient

An interesting question is how α changes with w? In order to answer this question we will need to know how to measure risk-aversion?

Definition 4.18 *Absolute RA coefficient:* $R_a = -\dfrac{u''(x)}{u'(x)} > 0$ *(Arrow-Pratt coefficient)*

For example, for $u(x) = \beta - \gamma e^{-ax}$ absolute risk aversion coefficient is constant. This family of utility functions is called CARA utility functions (constant absolute risk aversion) utility function. It is easy to see that the Arrow-Pratt coefficient is constant: $u'(x) = a\gamma e^{-ax}$, $u''(x) = -a^2\gamma e^{-ax}$, $R_a = -\dfrac{u''(x)}{u'(x)} = a$, CARA$\Rightarrow \dfrac{d\alpha}{dw} = 0$.

Claim 4.19 *If investors have CARA utility function then α doesn't change with the change of w.*

Proof. $FOC: U'(\alpha) = \Sigma\pi_i(x_i - \tau)u'(\alpha(x_i - \tau) + \tau w) = 0 = F(\alpha, w)$. By Implicit Function Theorem,

$$
\begin{aligned}
\frac{d\alpha}{dw} = -\frac{\partial F/\partial w}{\partial F/\partial \alpha} &= -\frac{\Sigma\pi_i(x_i - \tau)\tau \cdot u''(\alpha(x_i - \tau) + \tau w)}{\Sigma\pi_i(x_i - \tau)^2 u''(\alpha(x_i - \tau) + \tau w)} \\
&= \frac{\Sigma\pi_i(x_i - \tau)\tau \cdot \alpha u'(\alpha(x_i - \tau) + \tau w)}{\Sigma\pi_i(x_i - \tau)^2 u''(\alpha(x_i - \tau) + \tau w)} = 0.
\end{aligned}
$$

■

Thus investors with constant risk aversion always invest the same amount of money into the risky asset regardless of their wealth.

Definition 4.20 *Relative RA coefficient* $R_r = -x\dfrac{u''(x)}{u'(x)}$

For example, if $u(x) = \ln x$, then $R_r(x)$ is constant: $u'(x) = \dfrac{1}{x}$, $u''(x) = -\dfrac{1}{x^2}$, $R_r(x) = x \cdot \dfrac{1}{x^2} \cdot x = 1$. Such functions are called CRRA (constant relative risk-aversion). For CRRA-utility functions $\Rightarrow \dfrac{d(\alpha/w)}{dw} = 0$, that is proportion of α in w doesn't change with the change of w (Problem Set).

5 Production

5.1 Basic definitions

- *Firms:* buy inputs, transform them into outputs and maximize profit.
- *Production:* process of transforming inputs into outputs.
- *Technology:* determines restrictions on what is possible by production.

Assume a firm uses n inputs to produce 1 output. Technology can be described by a production function: $y = f(x_1, x_2, ..x_n)$. It is an increasing function with respect to each input x_i. We call $\dfrac{\partial f(x_1, x_2, ..x_n)}{\partial x_i} = MP_i$ marginal product of input i, which is similar to marginal utility MU_i. MRTS(marginal rate of technical substitution) is the absolute value of the slope of the isoquant.

Example 5.1 $f(K, L) = K^\alpha L^\beta = y$ *Cobb-Douglas production function,*

$$y = K^\alpha L^\beta$$
$$MP_K = \frac{\partial f}{\partial K} = \partial K^{\alpha-1} L^\beta$$
$$MP_L = \frac{\partial f}{\partial L} = \beta K^\alpha L^{\beta-1}$$
$$MRTS_{ij} = \frac{MP_i}{MP_j}$$
$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{\beta K^\alpha L^{\beta-1}}{\partial K^{\alpha-1} L^\beta} = \frac{\beta K}{\alpha L}$$

Definition 5.2 *We classify the production functions in the following way:*

- *if* $f(tx) = f(tx_1, \ldots, tx_n) = tf(x)$ $\forall x$ *and* $\forall t > 0$ *then* f *exhibits constant return to scale (CRS)*
- *if* $f(tx) = f(tx_1, \ldots, tx_n) > tf(x)$ $\forall x$ *and* $\forall t > 1$ *then* f *exhibits increasing return to scale (IRS)*
- *if* $f(tx) = f(tx_1, \ldots, tx_n) < tf(x)$ $\forall x$ *and* $\forall t > 1$ *then* f *exhibits decreasing return to scale (DRS)*

Example 5.3 $f(K, L) = K^\alpha L^\beta$,

$$f(tK, tL) = t^{\alpha+\beta} K^\alpha L^\beta = t^{\alpha+\beta} f(K, L)$$
$$\text{if } \alpha + \beta > 1 \Rightarrow \textit{ IRS}$$
$$\text{if } \alpha + \beta = 1 \Rightarrow \textit{ CRS}$$
$$\text{if } \alpha + \beta < 1 \Rightarrow \textit{ DRS}$$

5.2 Cost function

5.2.1 Long-run cost function

Assume that input prices are given $w = (w_1, w_2, ..w_n)$ by perfect competition market, that is the firm is small and is a price-taker. Then the firm cost-minimization problem is

$$\begin{cases} \min_x w_1x_1 + ... + w_nx_n \\ st. f(x_1, ..x_n) = y \end{cases}$$

$$L = w_1x_1 + ... + w_nx_n + \lambda(y - f(x_1, ..x_n))$$
$$\frac{\partial L}{\partial x_i} = w_i - \lambda\frac{\partial f(x)}{\partial x_i} = 0$$
$$\frac{\partial L}{\partial \lambda} = y - f(x_1, ..x_n) = 0$$
$$\Rightarrow \frac{w_i}{w_j} = \frac{MP_i}{MP_j} = MRTS_{ij}$$

Definition 5.4 *The input values that minimize the cost are called the conditional input demand $x(w, y)$. Notice that the conditional input demand depends on input prices and the desired output level, y. We call $\sum_i w_ix_i(y, w)$ a firm's cost function $c(w, y)$.*

Example 5.5 $f = K^\alpha L^\beta$

$$\begin{cases} \min_{K,L} w_KK + w_LL \\ st. K^\alpha L^\beta = y \end{cases}$$

$$\frac{w_K}{w_L} = \frac{\alpha L}{\beta K} \Rightarrow K = \frac{\alpha L}{\beta} \cdot \frac{w_K}{w_L}$$
$$(\frac{\alpha L}{\beta} \cdot \frac{w_K}{w_L})^\alpha L^\beta = y$$
$$\Rightarrow L = [\frac{y}{(\frac{\alpha}{\beta} \cdot \frac{w_L}{w_K})^\alpha}]^{\frac{1}{\alpha+\beta}}, K = [\frac{y}{(\frac{\beta}{\alpha} \cdot \frac{w_K}{w_L})^\beta}]^{\frac{1}{\alpha+\beta}}$$
$$c(w, y) = w_K \cdot [\frac{y}{(\frac{\beta}{\alpha} \cdot \frac{w_K}{w_L})^\beta}]^{\frac{1}{\alpha+\beta}} + w_L \cdot [\frac{y}{(\frac{\alpha}{\beta} \cdot \frac{w_L}{w_K})^\alpha}]^{\frac{1}{\alpha+\beta}}$$

Properties of cost function

1. $x(w, y)$ is homogeneous of degree 0 with respect to w: $x(tw, y) = x(w, y)$
2. $c(w, y)$ is homogeneous of degree 1 with respect to w: $c(tw, y) = tc(w, y)$

$$c(tw, y) = tw_1x(tw, y) + ... + tw_nx(tw, y)$$
$$= tw_1x(w, y) + ... + tw_nx(w, y) = tc(w, y)$$

3. $c(w, y)$ is increasing with respect to y and w.
4. $c(w, y)$ is concave in w.
5. Shephard Lemma: $\frac{\partial c(w,y)}{\partial w_i} = x_i(w, y)$.

Proofs of these properties are identical to the proofs of properties of the expenditure function.

We will also refer to $c(w, y)$ as long-run cost function. The reason is that the firm can choose different levels of inputs which is usually possible only in the long-run.

Definition 5.6 *Long-run average and long-run marginal costs are defined as follows:*

$$\begin{aligned} LRAC(w,y) &= \frac{c(w,y)}{y} \\ LRMC(w,y) &= \frac{\partial c(w,y)}{\partial y} \end{aligned}$$

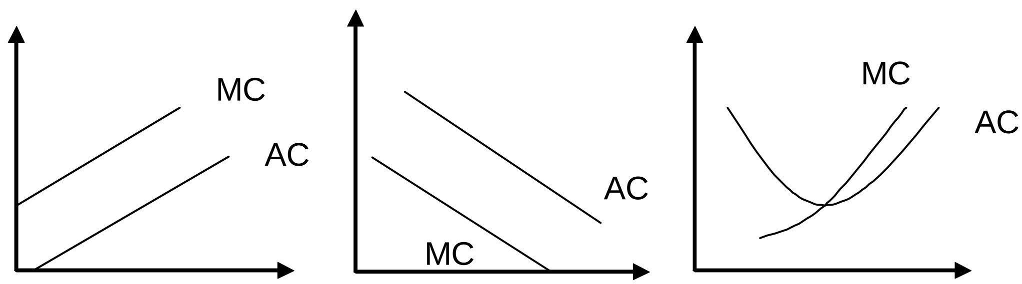
Claim 5.7 *If long-run Average Cost is increasing with respect to y, then $LRMC(y) > LRAC(y)$ If long-run Average Cost is decreasing with respect to y then $LRMC(y) < LRAC(y)$.*

Proof. We want to prove that $LRMC(y) > LRAC(y)$. By definition $LRAC(w,y) = \dfrac{c(w,y)}{y}$ so

$$\left(\frac{c(w,y)}{y}\right)' = \frac{c'(w,y)y - c(w,y)}{y^2} = \frac{1}{y}\Big(\underbrace{\frac{c'(w,y)}{y}}_{MC} - \underbrace{\frac{c(w,y)}{y}}_{AC}\Big) > 0$$

$$\Leftrightarrow MC > AC$$

■



5.2.2 Short-run cost function

In the short run firm cannot vary all of its inputs. Let us assume that in total there are n inputs and among them values of inputs $\overline{x}_{m+1}, ..\overline{x}_n$ are fixed. The cost-minimization problem is

$$\begin{cases} \min_{x_1,,x_m} w_1x_1 + ... + w_mx_m + w_{m+1}\overline{x}_{m+1} + .. + w_n\overline{x}_n \\ st. f(x) = y \end{cases}$$

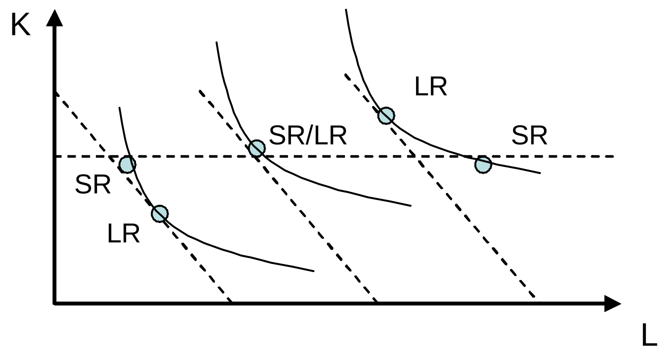
Example 5.8 *Let $f(K,L) = K^\alpha L^\beta$ so that inputs are $\overline{K}$ and L.*

$$\begin{cases} \min_L w_K\overline{K} + w_LL \\ st. \overline{K}^\alpha L^\beta = y \end{cases} \Rightarrow L = (\frac{y}{\overline{K}^\alpha})^{\frac{1}{\beta}},$$

Thus the short-run cost function is $SC(w,y,\overline{K}) = w_K\overline{K} + w_L(\frac{y}{\overline{K}^\alpha})^{\frac{1}{\beta}}$. Here $w_K\overline{K}$ is the fixed cost doesn't depend on y. Term $w_L(\frac{y}{\overline{K}^\alpha})^{\frac{1}{\beta}}$ is the variable cost that depends on y.

In the same way as we did for the long-run cost function we can define short-run marginal and average costs:

$$\begin{aligned} SRAC(w,y,\overline{K}) &= \frac{sc(w,y,\overline{K})}{y} \\ SRMC(w,y,\overline{K}) &= \frac{\partial sc(w,y,\overline{K})}{\partial y}. \end{aligned}$$



Also it is clear that $c(w,y) \leqslant sc(w,y,\overline{K})$ for any $\overline{K}$, since we have less degrees of freedom to minimize cost.

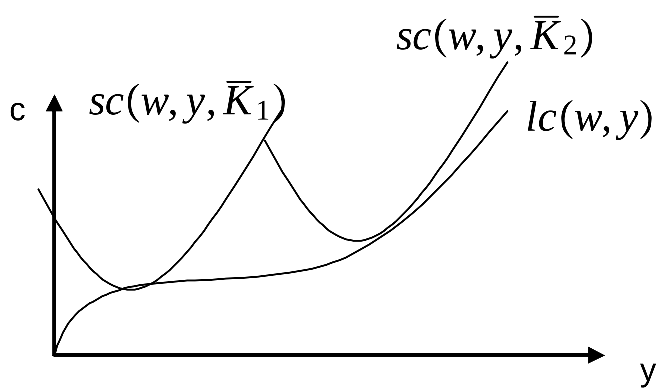
If $\overline{K} = K(w,y) \Rightarrow c(w,y) = sc(w,y,K(w,y))$ and so at this point short-run cost is minimal. That is

$$\frac{\partial sc(w,y,\overline{K})}{\partial \overline{K}} = 0 \text{ when } \overline{K} = K(w,y).$$

In addition we can see that

$$\frac{\partial c(w,y)}{\partial y} = \frac{\partial sc(w,y,K(w,y))}{\partial y} + \underbrace{\frac{\partial sc(w,y,K(w,y))}{\partial K}}_{=0} \cdot \frac{\partial K(w,y)}{\partial y},$$

where the second term is equal to zero, because of the FOC. Thus, we have that short-run cost is always greater than long-run cost and at levels where $SRC = LRC$, they are tangent as shown on the picture.



5.3 Profit function

Assume that firm is small so that the output price doesn't depend on firm's behavior. Then given the input prices $w = (w_1, w_2, ..w_n)$ firm's maximization problem is

$$\begin{cases} \max_{x,y} py - \Sigma w_i x_i \\ st. y = f(x) \end{cases} \Leftrightarrow \max_{x,y} pf(x) - \Sigma w_i x_i$$

$$FOC : p\frac{\partial f(x)}{\partial x_i} - w_i = 0 \; p\frac{\partial f(x)}{\partial x_j} - w_j = 0$$
$$\Rightarrow \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j} = \frac{w_i}{w_j} = MRTS_{ij}$$
$$\Rightarrow x(p, w)$$

Solution to this problem is $x(p, w)$ — *input demand.* The difference from conditional input demand $x(y, w)$ is that $x(y, w)$ depends on y so that $x(y, w)$ is the input demand condition on the output level. The output produced by the firm in the optimum — $y(p, w) = f(x(p, w))$— is called *output supply function.*

Obviously, $x(y(p, w), w) = x(p, w)$, that is conditional input demand=input demand. This is because they satisfy the same FOC and $y = f(x)$.

Note that $\pi(p, w)$ may be not defined in each situation. For example, when $f(x)$ has IRS, assume that $\exists x', y'$ that maximize profit, and optimal profit is nonnegative, then

$$pf(tx') - t\Sigma w_i x_i' \geqslant ptf(x') - t\Sigma w_i x_i' = t(pf(x') - \Sigma w_i x_i') = t\pi(p, w)$$

The firm could always increase its profit by demanding tx', and thus the firm would have infinite demand for inputs and would produce infinite output. Similarly, if $f(x)$ has CRS then unless the optimal profit is zero we run into the same problem:

$$pf(tx') - t\Sigma w_i x_i' = t(pf(x') - \Sigma w_i x_i') = t\pi(p, w) > \pi(p, w)$$

From now on assume that $\pi(p, w)$ is well defined.

Properties of profit function:

1. π is increasing with respect to p.
2. π is decreasing with respect to w.
3. π is homogeneous function of degree 1 in (p, w) that is $\pi(\lambda p, \lambda w) = \lambda\pi(p, w)$.
4. Hotelling Lemma.
$$\frac{\partial \pi(p, w)}{\partial p} = y(p, w) \text{ and } -\frac{\partial \pi(p, w)}{\partial w_i} = x_i(p, w).$$

The first equation is $\geqslant 0$, which is the proof of 1. The second equation is $\leqslant 0$, which is the proof of 2.

5. $\pi(p, w)$ is convex in (p, w).

Proof. Assume that given (p^0, w^0), the firm chooses (y^0, x^0) and given (p^1, w^1), the firm chooses (y^1, x^1). We want to prove that

$$\pi(\lambda p^0 + (1-\lambda)p^1, \lambda w^0 + (1-\lambda)w^1) \leqslant \lambda\pi(p^0, w^0) + (1-\lambda)\pi(p^1, w^1) \; \forall \lambda \in [0, 1].$$

Indeed,
$$\pi(p^0, w^0) = p^0 y^0 - w^0 x^0 \geqslant p^0 y^\lambda - w^0 x^\lambda$$
because (y^0, x^0) maximizes profit given (p^0, w^0). Similarly,
$$\pi(p^1, w^1) = p^1 y^1 - w^1 x^1 \geqslant p^1 y^\lambda - w^1 x^\lambda$$
From these two inequalities we can get
$$\lambda\pi(p^0, w^0) + (1-\lambda)\pi(p^1, w^1) \geqslant \pi(\lambda p^0 + (1-\lambda)p^1, \lambda w^0 + (1-\lambda)w^1) = p^\lambda y^\lambda - w^\lambda x^\lambda$$

■

Example 5.9 $f(K,L) = K^{\alpha}L^{\beta} \quad \alpha + \beta \leqslant 1$

$$\max_{K,L} \pi(p,w) = pK^{\alpha}L^{\beta} - w_K K - w_L L$$
$$FOC : \alpha p K^{\alpha-1}L^{\beta} = w_K$$
$$\beta p K^{\alpha} L^{\beta-1} = w_L$$

In short-run, capital is fixed at $\overline{K}$.

$$\max_{L} \pi(p,w) = p\overline{K}^{\alpha}L^{\beta} - w_K\overline{K} - w_L L$$
$$FOC : \beta p K^{\alpha} L^{\beta-1} = w_L$$

Then,

$$L^* = \left(\frac{w_L}{\beta p K^{\alpha}}\right)^{\frac{1}{\beta-1}} = w_L^{\frac{1}{\beta-1}} \cdot \beta^{\frac{1}{1-\beta}} \cdot p^{\frac{1}{1-\beta}} \cdot (\overline{K}^{\alpha})^{\frac{1}{1-\beta}} = w_L^{\frac{1}{\beta-1}} \cdot \beta^{\frac{1}{1-\beta}} \cdot p^{\frac{1}{1-\beta}} \cdot \overline{K} \quad (\alpha = 1-\beta)$$

So the short-run profit is

$$SR\pi(p, w_L, w_K, \overline{K}) = w_L^{\frac{\beta}{\beta-1}} \cdot \beta^{\frac{\beta}{1-\beta}} \cdot p^{\frac{1}{1-\beta}} \cdot \overline{K} - w_K\overline{K}.$$

Finally, we can derive a supply function in two ways. Either plug L^* into the production function, or use the Hotelling Lemma that is $\dfrac{\partial \pi}{\partial p} = y(p,w)$.

We conclude this section with the observation that the short-run profit can be negative. Indeed,

$$SR\pi(p, w_L, w_K, \overline{K}) = \max_{y} py - SC(y, w_L, w_K, \overline{K})$$
$$FOC : p = \frac{\partial SC(y)}{\partial y} = SRMC(y).$$

Suppose now that the solution $y_1 > 0$ and so in order to find the optimal production level we need to compare $SR\pi(y_1)$ with $SR\pi(0)$. $SR\pi(0) = 0 - w_K\overline{K} - 0 = -w_K\overline{K} < 0$. $SR\pi(y_1)$ can be negative, for example if $w_K\overline{K}$ is huge.

$$SR\pi(y_1) = py_1 - FC - VC(y_1)$$
$$SR\pi(0) = 0 - FC$$

Thus,

$$\pi(y_1) > \pi(0) \Leftrightarrow py_1 - VC(y_1) > 0 \Leftrightarrow p > \frac{VC(y_1)}{y_1}$$

6 Partial equilibrium

There are two elements in partial equilibrium. The first one we consider a market for only one good: input prices are fixed and prices for other commodities are also fixed. The second: the market is perfectly competitive, that is firms and companies are price-takers.

Consumers behavior is determined by utility-maximization. In particular, from this we derive their demand functions. Assume we have I consumers with demand: $q^i(p, \widetilde{p}, y^i) = q^i(p)$. Here p is the price of the good, $\widetilde{p}$ is the price of other goods, and y^i is the income of consumer i. The total demand is

$$\Sigma q^i(p) = q^D(p)$$

Firms maximize profits, from which we can derive their supply functions. Assume there are J firms: $q^j(p, w) = q^j(p)$. Since w is fixed we drop it from the notations. Total supply is

$$\Sigma q^j(p) = q^S(p)$$

In the equilibrium

$$q^D(p) = q^S(p)$$

Example 6.1 *In short-run, suppose there are 48 identical firms with Cobb-Douglas production function $f(\overline{K}, L) = \overline{K}^{1-\alpha} L^{\alpha}$, let $\alpha = \frac{1}{2}, w_L = 4, w_K = 1, \overline{K} = 1$. We already know that*

$$q^j(p) = w_L^{\frac{\alpha}{\alpha-1}} \cdot \alpha^{\frac{\alpha}{1-\alpha}} \cdot p^{\frac{\alpha}{1-\alpha}} \cdot \overline{K}$$

So $q^j(p) = \dfrac{p}{8}$, and $q^S(p) = 48(\dfrac{p}{8}) = 6p$. Assume the total demand is $q^D(p) = \dfrac{294}{p}$. In the equilibrium:

$$\frac{294}{p} = 6p \Rightarrow p_{eq} = 7, q_{eq} = 42, q^j(p_{eq}) = \frac{7}{8}$$

$$SR\pi^j(p_{eq}) = w_L^{\frac{\beta}{\beta-1}} \cdot \beta^{\frac{\beta}{1-\beta}} \cdot p^{\frac{1}{1-\beta}} \cdot \overline{K} - w_K \overline{K} = \frac{33}{16}$$

The difference between short-run and the long-run is that in the short-run number of firms is fixed. On the other hand in the long-run firms can enter or exit the market until each firm earns zero profit (otherwise if $\pi > 0$, new firms will enter; if $\pi < 0$, firms will exit). This gives us a condition on number of firms in the long-run. To sum up we find market equilibrium in the long-run and in the short-run differently:

$$SR \begin{cases} \Sigma q^j(p) = \Sigma q^i(p) \\ J \text{ is given} \end{cases} \qquad LR \begin{cases} \Sigma q^j(p) = \Sigma q^i(p) \\ \pi^j(p_{eq}) = 0 \end{cases}.$$

Example 6.2 *Assume total demand is $q^D(p) = 230 - 30p$; firm's profit is $\pi^j(p) = \dfrac{16}{p^2} + \dfrac{35}{8}p - \dfrac{375}{16}$. Then from Hotelling Lemma, the supply function is $q^j(p) = \dfrac{\partial \pi^j(p)}{\partial p} = \dfrac{1}{8}(p + 35)$. In the long-run*

$$\pi^j(p_{eq}) = 0 \Rightarrow p_{eq} = 5$$

Then $q^D(p_{eq}) = 80, q^j(p_{eq}) = 5 \Rightarrow J = \dfrac{80}{5} = 16$.

In long-run, from FOC $P = MC(q^j)$. From zero-profit condition that firm j produces q^j when $AC(q^j) = MC(q^j)$, because

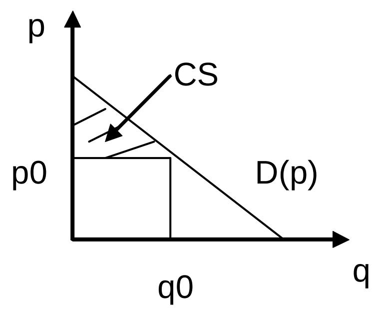
$$\pi^j(p) = pq^j - c(q^j) = 0,\ p = \frac{c(q^j)}{q^j} = AC(q^j)$$

6.1 Efficiency of competitive outcome

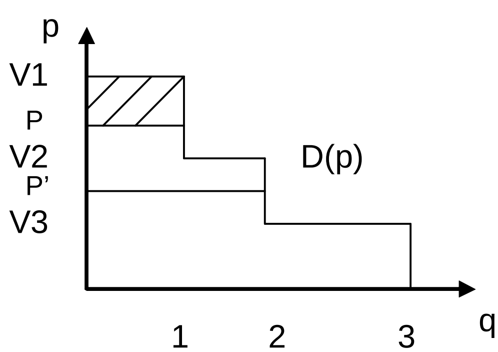
Definition 6.3 *Whenever it is possible to make someone better off without make any one else worse off, we say that Pareto improvement can be made.*

Definition 6.4 *If there is no way to make Pareto improvement then the situation is Pareto efficient.*

Suppose there are only one consumer and one firm in competitive market. Consumer surplus at p_0, q_0 is an area below Marshland demand curve and above p_0. Consumer surplus at p_0, q_0 is a measure of how much money consumer is willing to pay for the right of buying q_0 units good at price p_0.



Example 6.5 *Suppose a consumer's willingness to pay for ice-cream is $V_i = 5\$$, the market price of ice-cream is $p_j = 2\$$, so $CS = 3\$$,. Now suppose that V_1 is the value of the first unit of ice-cream,V_2 is the valuation of the second unit of ice-cream, and V_3 is valuation of the third unit of ice-cream. Then*

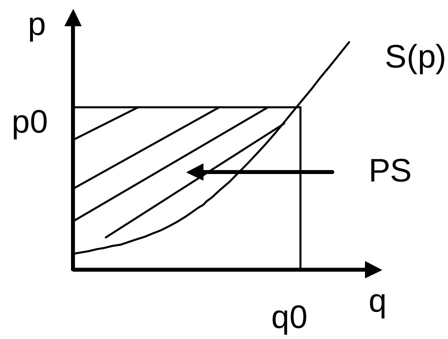


$$CS_1 = (V_1 - p) \cdot 1$$
$$CS_2 = (V_1 - p') \cdot 1 + (V_2 - p') \cdot 1$$

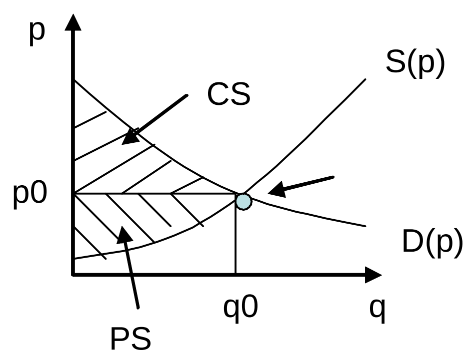
Producer Surplus(PS)= revenue–total variable cost (we neglect fixed cost here). If fixed cost is equal to zero, then

$$PS = \text{firms profits}$$

If we depict $p = MC(q)$ as the supply curve, and revenue=p_0q_0, then we can get that $PS = p_0q - \int\limits_0^{q_0} MC(q)dq$ and $TVC = \int\limits_0^{q_0} MC(q)dq = C(q_0) - C(0)$, as the following picture shows:



We can get Pareto efficient from the intersection of supply curve and demand curve as following picture shows:



6.2 Tax incidence

Assume the market demand function is $q^D(p)$, and the market supply function is $q^S(p)$. Consumer has to pay tax t for each unit he buys, that is the price he pays is $p^{eq}+t$. Then the new equilibrium price is a function of t:

$$q^D(p+t) = q^S(p)$$

Now, let's consider the elasticity of price (η). By FOC,

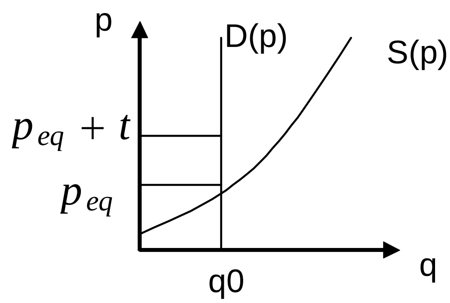
$$\frac{\partial q^D(p+t)}{\partial p} \cdot (\frac{\partial p}{\partial t} + 1) = \frac{\partial q^S(p)}{\partial p} \cdot \frac{\partial p}{\partial t}$$
$$\Rightarrow \frac{\partial p}{\partial t} = \frac{\partial q^D(p+t)/\partial p}{\partial q^S(p)/\partial p - \partial q^D(p)/\partial p} < 0$$
$$= \frac{\partial q^D(p)/\partial p \cdot p/q^D}{\partial q^S(p)/\partial p \cdot p/q^S - \partial q^D(p)/\partial p \cdot p/q^D}$$
$$= \frac{\eta_D}{\eta_S - \eta_D} < 0$$

Thus whether in equilibrium the price will increase or decrease as we introduce tax depends on elasticities of demand and supply functions.

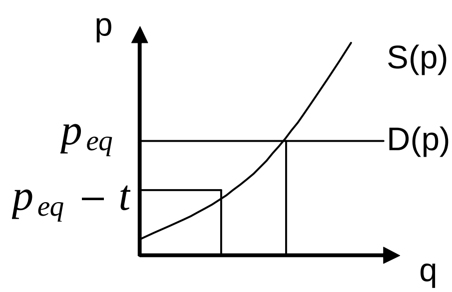
The price elasticity of consumer is

$$\frac{\partial p^D}{\partial t} = \frac{\partial(p+t)}{\partial p} = \frac{\eta_D}{\eta_S - \eta_D} + 1 = \frac{\eta_S}{\eta_S - \eta_D} > 0$$

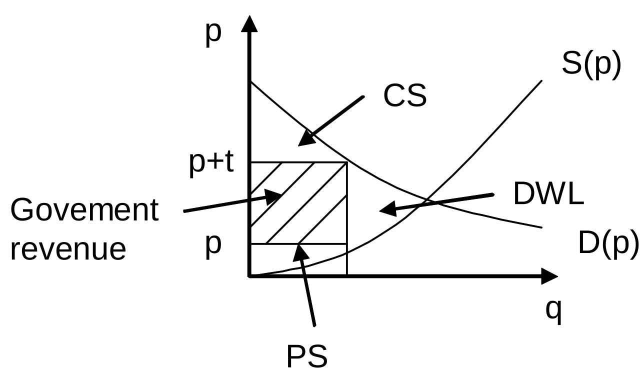
Example 6.6 *Assume $\eta_D = 0$, then $\frac{\partial p}{\partial t} = 0$ and $\frac{\partial p^D}{\partial t} = 1$, so consumers pay tax in this condition.*



Assume $\eta_D = \infty$, then $\frac{\partial p}{\partial t} = -1$ and $\frac{\partial p^D}{\partial t} = 0$, so firms pay tax in this condition.



Assume consumers and firms share the tax together at last, then



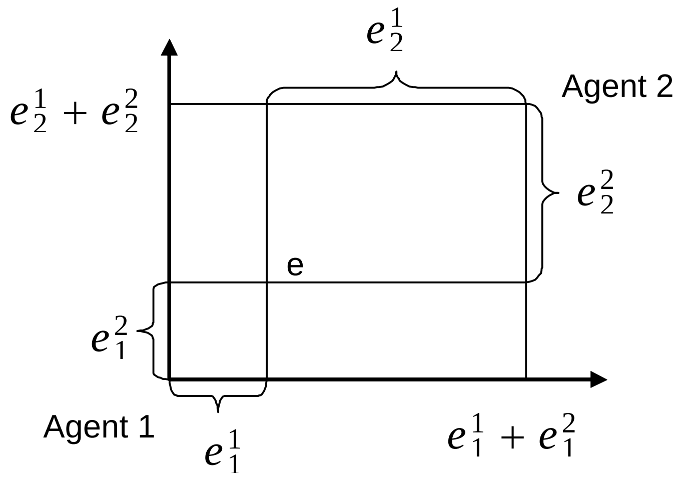
7 General equilibrium

Suppose there are I individuals, N commodities, and no production in the economy. Each individual has $\succcurlyeq$ and initial endowment, e_n^i, where i stands for individual and n for commodity. Let $e^1 = (e_1^1, e_2^1)$,

$e^2 = (e_1^2, e_2^2)$, then

$$e = e^1 + e^2 = (e_1^1 + e_1^2, e_2^1 + e_2^2).$$

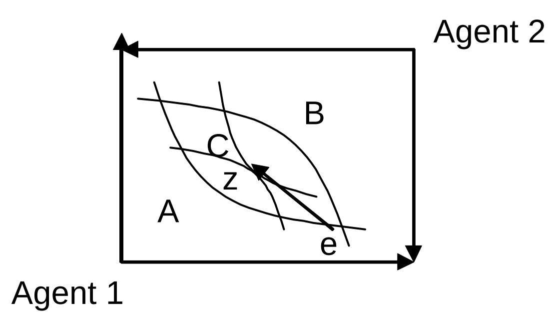
We can use Edgeworth box to depict the above conditions. Each input in the Edgeworth box represents a feasible allocation.



If we define allocation as $x = (x^1, x^2)$, then the feasible allocation is $x^1 + x^2 = e^1 + e^2$, that is

$$\begin{cases} x_1^1 + x_1^2 = e_1^1 + e_1^2 \\ x_2^1 + x_2^2 = e_2^1 + e_2^2 \end{cases}$$

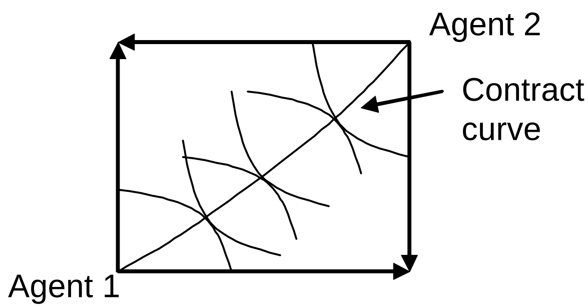
Assume agents have e initial endowment; there is no money and no market; only goods exchange can take place and the exchange is voluntary here. Then what is the outcome?



As we can see in the above picture, first, the exchange cannot end up in A area or B area. This is because one guy will be worse off in that condition and will not be willing to trade. No one will refuse to move to C area. Second, if they end at point y, they still have incentive to trade. Finally, at point z, indifference curve are tangent, so they don't trade.

Definition 7.1 *An allocation inside the Edgeworth box is Pareto efficient if there is no other feasible allocation $y = (y^1, y^2)$ such that $y^1 \succcurlyeq x^1, y^2 \succcurlyeq x^2$, with at least one inequality strict.*

Definition 7.2 *A set of all Pareto efficient allocations in the Edgeworth box is called contract curve.*



Note that starting from e will finish on a contract curve which is between the indifference curves passing through e.

Now assume there are I consumers, N goods. The initial endowment is n vector $e = (e^1, , e^I)$, allocation is $x = (x^1, , x^I)$, and the specified bundle for each i consumer is $x^i = (x_1^i, , x_N^i)$.

Definition 7.3 *An allocation is feasible if* $\forall n\ \Sigma x_n^i = \Sigma e_n^i$.

Definition 7.4 *An allocation x is Pareto efficient if there is no other feasible allocation y, so that $y^i \succcurlyeq^i x^i$ $\forall i$ and at least one is strict. Without markets, consumers will end up on a Pareto efficient allocation.*

7.1 Equilibrium in competitive markets

Assume all agents are small, so they do not affect the price. Assume also that agent preferences are continuous, monotone and convex.

Denote price for N goods as $p = (p_1, , , p_N)$. If consumer i with initial endowment e^i ends up with a bundle x^i, then there are three possibilities:

- $x_n^i - e_n^i > 0$ that is i buys $x_n^i - e_n^i$ units of good n and pays $p_n(x_n^i - e_n^i)$;
- $x_n^i - e_n^i < 0$ that is i sells $x_n^i - e_n^i$ of good n and receives $p_n(x_n^i - e_n^i)$;
- $x_n^i - e_n^i = 0$ that is i just consumes his endowment of good n.

The budget constraint for a consumer with endowment e^i and prices p is that money spend on trade are less or equal than money received from the trade. Assume that preferences are monotone and then we can write it as

$$\underset{x_n^i > e_n^i}{\Sigma} p_n(x_n^i - e_n^i) = \underset{x_n^i < e_n^i}{\Sigma} p_n(e_n^i - x_n^i) \Rightarrow \Sigma p_n x_n^i = \Sigma p_n e_n^i$$

Informally, in the equilibrium

i) each consumer should maximize his utility given his endowment and prices;

ii) total supply should be equal to total demand.

Definition 7.5 *Excess demand for good n is total demand for good n minus total supply of good n that is*

$$z_n(p) = \Sigma(x_n^i(p, p \cdot e^i) - e_n^i) = \underbrace{\Sigma x_n^i(p, p \cdot e^i)}_{TD} - \underbrace{\Sigma e_n^i}_{TS}$$

If $z_n(p) > 0$, then $TD > TS$; if $z_n(p) < 0$, then $TD < TS$; if $z_n(p) = 0$, then market clears.

Definition 7.6 *The aggregate excess demand function is* $z(p) = (z_1(p), , , z_n(p))$.

Properties of $z(p)$:
1. $z(p)$ is continuous, because demand is continuous.
2. $z(p)$ is homogeneous of degree 0 in p.

$$z_n(\lambda p) = \Sigma x_n^i(\lambda p, \lambda p \cdot e^i) - \Sigma e_n^i = \Sigma x_n^i(p, p \cdot e^i) - \Sigma e_n^i = z_n(p)$$

3. Walras law: for any p,

$$p \cdot z(p) = 0 \Leftrightarrow p_1 z_1(p) + p_2 z_2(p) + .. + p_n z_n(p) = 0$$

Proof. Because budget consumption should hold with equality, then

$$\Sigma_{n=1}^N p_n(x_n^i(p, p \cdot e^i) - e_n^i) = 0$$
$$\Sigma_{i=1}^I \Sigma_{n=1}^N p_n(x_n^i(p, p \cdot e^i) - e_n^i) = 0$$
$$\Sigma_{n=1}^N p_n \Sigma_{i=1}^I (x_n^i(p, p \cdot e^i) - e_n^i) = 0$$
$$\Rightarrow \Sigma_{n=1}^N p_n z_n(p) = 0$$

■

Example 7.7 *Assume* $N = 2$, *from Walras law we know that:* $p_1 z_1(p) + p_2 z_2(p) = 0$, *if* $p_1 z_1(p) > 0 \Rightarrow p_2 z_2(p) < 0$; *if* $p_1 z_1(p) < 0 \Rightarrow p_2 z_2(p) > 0$; *if* $p_1 z_1(p) = 0 \Rightarrow p_2 z_2(p) = 0$. *Generally, we can get*

$$p_1 z_1(p) + p_2 z_2(p) + .. + p_n z_n(p) = 0$$

so

$$z_1(p) = z_2(p) = .. = z_{n-1}(p) \Rightarrow z_n(p) = 0.$$

Definition 7.8 *In competitive market,* $p^* \in R_{++}^N$ *is called a Walrasian equilibrium price if* $z(p^*) = 0$.

How to find Walrasian equilibrium? Given $\succcurlyeq^i, e^i$, we can find $x^i(p, p \cdot e^i) = (x_1^i(p, p \cdot e^i), , , x_n^i(p, p \cdot e^i))$ for each consumer and then derive $z(p) = (z_1(p), , , z_n(p))$. Having found $z(p)$ we need to find p^* such that $z(p^*) = 0$.

Notice that from the Walras law it follows that the system $z(p) = 0$ has $n-1$ independent equations thus we have one degree of freedom. In particular, we can assume that $p_1^* = 1$. Having found the equilibrium price we can find *equilibrium allocation*: $x^* = (x^1(p^*, p^* \cdot e^i), x^2(p^*, p^* \cdot e^i), , x^I(p^*, p^* \cdot e^i))$.

Example 7.9 *Assume* $I = 2, N = 2$, *the initial endowment for each agent is* $e^1 = (1, 0), e^2(0, 1)$, *and the utility function for them is* $u^1(x_1, x_2) = u^2(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$.

For the first agent, $\begin{cases} \max_{x_1, x_2} \sqrt{x_1} + \sqrt{x_2} \\ p_1 x_1 + p_2 x_2 = p_1 \cdot 1 + p_2 \cdot 0 \end{cases}$, *thus the Lagrange function is*

$$L = \sqrt{x_1} + \sqrt{x_2} + \lambda(p_1 - p_1 x_1 - p_2 x_2)$$
$$FOC : \frac{\partial L}{\partial x_1} = \frac{1}{2\sqrt{x_1}} - \lambda p_1 = 0, \frac{\partial L}{\partial x_2} = \frac{1}{2\sqrt{x_2}} - \lambda p_2 = 0$$
$$\Rightarrow x_2 = (\frac{p_1}{p_2})^2 x_1$$

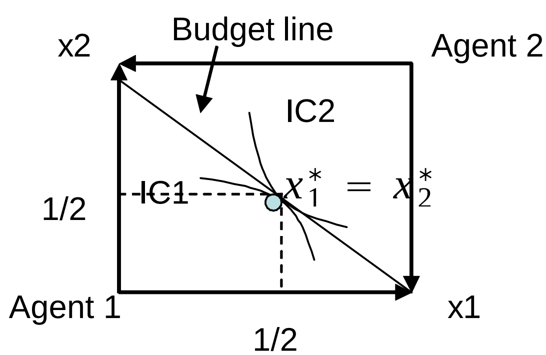
Plugging x_2 into budget constraint,

$$p_1x_1 + p_2(\frac{p_1}{p_2})^2x_1 = p_1$$
$$\Rightarrow x_1^1 = \frac{p_2}{p_1+p_2}, x_2^1 = \frac{p_1}{p_1+p_2}$$

For the second agent, $\begin{cases} \max_{x_1,x_2} \sqrt{x_1} + \sqrt{x_2} \\ p_1x_1 + p_2x_2 = p_1 \cdot 0 + p_2 \cdot 1 \end{cases}$ *, from FOC we get* $x_1^2 = \frac{p_2}{p_1} \cdot \frac{p_2}{p_1+p_2}$, $x_2^2 = \frac{p_1}{p_1+p_2}$.
For good 1, let $p_1^* = 1$,

$$z_1(p) = \frac{p_2}{p_1+p_2} + \frac{p_2}{p_1} \cdot \frac{p_2}{p_1+p_2} - 1 = 0$$
$$\Leftrightarrow z_1(p) = \frac{p_2}{1+p_2} + \frac{p_2}{1} \cdot \frac{p_2}{1+p_2} - 1 = 0$$
$$\Rightarrow p_2^* = 1$$

So the equilibrium price is $p_1^* = p_2^* = 1$, *Walrasian allocation is* $x_1^*(p^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $x_2^*(p^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$.



Theorem 7.10 *Assume that consumer's utility are continuous, strict increasing and strictly quasiconcave. Assume also that* $\Sigma e^i >> 0$. *Then* $\exists p^*$ *such that* $z(p^*) = 0$ *that is Walrasian Equilibrium.*

8 General Equilibrium and Welfare

Definition 8.1 *An allocation x is Pareto efficient if there is no other feasible allocation y, so that* $y^i \succcurlyeq^i x^i$ $\forall i$ *with at least one preference being strict.*

Theorem 8.2 (First Welfare Theorem) *Any competitive equilibrium allocation is PE.*

Proof. Assume not. Let (p, x) be a WE and assume there exists feasible y such that $y^i \succcurlyeq x^i$ (with at least one strict). Take person i for whom $y^i \succ x^i$. Then $py^i > px^i$ (why?). For any person j for whom $y^j \succcurlyeq x^j$ it has to be the case that $py^j \geqslant px^j$. Then we can get

$$p\Sigma e^j = p\Sigma y^j > p\Sigma x^j = p\Sigma e^j,$$

which is a contradiction. ■

We have just established that any WE is PE however just Pareto efficiency might be not enough. As we remember Pareto Efficiency has nothing to do with fairness. In particular it could be possible that in the equilibrium allocation one agent receives almost everything and all other agents receive almost nothing. While this is still Pareto efficient the government may want to interfere and try to achieve a resource allocation that is more even. At the same time the government does not want to sacrifice the efficiency. As the Second Welfare Theorem shows any Pareto efficient allocation can be achieved as a WE after appropriate transfers. For example, if there is a PE allocation that the government (for whatever reasons) believes is more fair then it can be achieved in equilibrium after some transfers.

In the proof of the SWT we will use the following result.

Theorem 8.3 (Separating Hyperplane Theorem) *A convex set* $A \subset \mathbb{R}^n, w \in \mathbb{R}^n, w \notin int(A)$, *then* $\exists p \in \mathbb{R}^n$ *st* $pz > pw$ *for any* $z \in int(A)$. *The theorem can be shown by the following picture.*

Theorem 8.4 (Second Welfare Theorem) *Assume that preferences of all agents in the economy are strictly convex. Let* x *be a PE allocation, then* $\exists p$ *st.*(p, x) *is a WE of the economy with endowment* x.

In other words, FWT says that any WE is PE. The SWT says that any PE allocation can be achieved as WE after transfers.

Proof. Suppose x is PE allocation. Let $P^i = \{y^i \in \mathbb{R}^N, y^i \succcurlyeq^i x^i\}$. It is easy to see that P^i is convex because preferences are strictly convex. Define $P = \{y \in \mathbb{R}^N, y = \Sigma y^i$, when $y^i \in P^i\}$.

Step 1: P is a convex set. Indeed, since $y \in P$ it has to be that $y = \Sigma y^i, y^i \in P^i$. Similarly, since $y' \in P$ we have that $y' = \Sigma y^{i\prime}, y^{i\prime} \in P^i$. Then $\lambda y + (1-\lambda)y' = \sum_i y^{i\lambda} \in P$ where $y^{i\lambda} = \lambda y^i + (1-\lambda)y^{i\prime} \in P^i$.

Step 2: Given that $w = \Sigma e^i = \Sigma x_i$, we will show that $w \notin int(P)$. Indeed, since x is PE if $w \in int(P)$ then

$$\begin{gathered} w = \lambda y + (1-\lambda)y' \\ y = \Sigma y^i, y^i \succcurlyeq^i x^i \\ y' = \Sigma y^{i\prime}, y^{i\prime} \succcurlyeq^i x^i \\ \Rightarrow y^{i\lambda} = \lambda y^i + (1-\lambda)y^{i\prime} > x^i \end{gathered}$$

Thus $y = (y^{1\lambda}, y^{2\lambda}, , , y^{I\lambda})$ is feasible, because $\Sigma y^{i\lambda} = \lambda \Sigma y^i + (1-\lambda)\Sigma y^{i\prime} = w$, and then x^i could not be PE. This proves that $w \notin int(P)$. However, $w \in P$, because $w = \Sigma x^i$, and $x^i \succcurlyeq^i x^i \Rightarrow x^i \in P^i$.

Step 3: By the Separating Hyperplane Theorem $\exists p$ such that $pz > pw, \forall z \in int(P)$. It can be shown that $p >> 0$. We do not provide a formal proof but the idea is that if $p_j \leq 0$ then $\tilde{x}^i_j \to \infty$ for any such j. This would involve a contradiction because on one hand consumer's demand $\tilde{x}$ would satisfy the budget constraint (since price of good j is non-positive) at the same time the consumer's demand $\tilde{x} \in P$ and by the Separating Hyperplane Theorem it should be that $p\tilde{x} > pw$.

Step 4: Given that $p >> 0$ it can be shown that (p, x) is a WE. Assume not, then $\exists i, y^i$ st. $y^i > x^i$ and $py^i = px^i$ (i.e. y^i is affordable). Let

$$y^j = x^j, \forall j \neq i \text{ and } z = \sum_{j=1}^{n} y^j.$$

Since $py^j = px^j$ it has to be the case that $pz = pw$.

The last step is to show that $z \in int(P)$ which will be a contradiction to the fact that $pz = pw$. However, this is obvious because $y^i \succ x^i$ and so $y^i \in int(P^i)$ and $y^j \in P^j$ for $j \neq i$. ∎

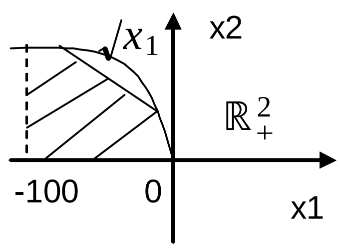
9 General equilibrium with production

There are two new issues in general equilibrium with production. The first one is distributing firm's profit across consumers. The second one is that since input of one firm can be output of another. Consequently, we cannot classify goods as inputs and outputs. We adopt the following sign convention: if a sign is less than zero, then firm uses good as an input; if a sign is greater than zero, then the firm uses the good as an output.

Suppose there are J firms, $y^j \in \mathbb{R}^N$ is a production plan of firm j, and $Y^j \subset \mathbb{R}^N$ is a set of all production plans.

Example 9.1 $Y = \{(-x_1, -x_2, x_3) : 0 \leqslant x_1 \leqslant 100, 0 \leqslant x_2 \leqslant 100, 0 \leqslant x_3 \leqslant f(x_1, x_2)\}$

$Y = \{(-x_1, x_2) : 0 \leqslant x_1 \leqslant 100, 0 \leqslant x_2 \leqslant \sqrt{x_1}\}$



Assumption on Y^j.

1. $0 \in Y^j$, means firm can decide to produce nothing.
2. $Y^j \cap \mathbb{R}^N_+ = \{0\}$, means to produce output you need inputs.
3. Y^j is closed and bounded.
4. Y^j is strictly convex.

The last assumption rules out IRS and CRS.

As before firm j chooses $y^j \in Y^j$ to max its profit. Let $p >> 0, y^j = (y^j_1, , , y^j_N)$, and $py^j = p_1 y^j_1 + p_2 y^j_2 + .. + p_N y^j_N$. If $y^j_N < 0$, then $p_N y^j_N$ is a part of a firm's cost; if $y^j_N > 0$, then $p_N y^j_N$ is a part of a firm's revenue. The firm's problem is

$$\max_{y^j \in Y^j} p \cdot y^j$$

The solution to this problem is $y^j(p)$ which is both output supply and input demands. When $y^j(p) > 0$, it's output, when $y^j(p) < 0$, it's input. Given $y^j(p)$, the profit function of firm j is

$$\pi^j(p) = p \cdot y^j(p).$$

Obviously, $y^j(p)$ is homogeneous of degree 0 with respect to price, and $\pi^j(p)$ is homogeneous of degree 1 with respect to price:

$$\pi^j(tp) = tp \cdot y^j(tp) = tp \cdot y^j(p) = t \cdot \pi^j(p).$$

Assume there are I consumers, and θ^{ij} is a share of consumers i in firm j so that $0 \leqslant \theta^{ij} \leqslant 1$ and $\sum_i \theta^{ij} = 1$. The consumer's budget constraint is

$$px^i \leqslant pe^i + \sum_{j=1}^{J} \theta^{ij} \pi^j(p)$$

The right side of this inequality is homogeneous of degree 1 because $\pi^j(p)$ is homogeneous of degree 1. The consumer problem is

$$\begin{cases} \max u(x^i) \quad x^i \in \mathbb{R}^n_+ \\ st.\ px^i \leqslant pe^i + \sum\limits_{j=1}^{J} \theta^{ij}\pi^j(p) \end{cases}$$

Consumer demand is $x^i(p)$, which is homogeneous of degree 0.

Definition 9.2 *Excess demand for good n is*

$$z_n(p) = \sum_i x^i_n(p) - \sum_i e^i_n - \sum_i y^j_n(p).$$

Excess demand is a vector $z(p) = (z_1(p), z_2(p), , z_n(p))$ and it is homogeneous of degree 0 *with respect to price.*

Claim 9.3 (Walras law) $\forall p >> 0, p \cdot z(p) = 0$

Proof. From the budget constraint,

$$px^i - pe^i - \sum_{j=1}^{J} \theta^{ij}\pi^j(p) = px^i - pe^i - \sum_{j=1}^{J} \theta^{ij}py^j(p) = 0$$

Sum it up for all i, and change the order of summation,

$$\sum_{i=1}^{I} px^i - \sum_{i=1}^{I} pe^i - \sum_{i=1}^{I}\sum_{j=1}^{J} \theta^{ij}py^j(p) = 0,$$
$$p\sum_{i=1}^{I} x^i - p\sum_{i=1}^{I} e^i - p\sum_{j=1}^{J}\sum_{i=1}^{I} \theta^{ij}y^j(p) = 0,$$
$$\sum_{i=1}^{I} x^i - \sum_{i=1}^{I} e^i - \sum_{j=1}^{J}\sum_{i=1}^{I} \theta^{ij}y^j(p) = 0,$$
$$\Rightarrow p \cdot z(p) = 0.$$

■

Definition 9.4 *Consider an economy $(u^i, e^i, \theta^{ij}, y^j)$, where $i = 1..I$, and $j = 1..J$. Price vector p^* is an equilibrium price vector if $z^*(p) = 0$.*

On one hand a system $z(p) = 0$ is a system of N equations and N unknowns. However, from Walras law, if $z^*_1(p) = 0, , , z^*_{n-1}(p) = 0$, then $z^*_n(p) = 0$, and so the last equations is redundant. At the same time we know that $z(p)$ is homogeneous of degree 0, so if $z^*(p) = 0$, then $z^*(tp) = 0$. Consequently when looking for a WE we can set a price of one good to 1, for example, $p_1 = 1$.

Given an equilibrium price p^* $x^* = (x^1(p^*), , x^I(p^*))$; $y^* = (y^1(p^*), , y^I(p^*)), (x^*, y^*)$ is a WE allocation.

Definition 9.5 *A Walrasian equilibrium is a triple (p, x, y) such that p is an equilibrium price vector, x is consumer's demand given p, and y is producer's output supply and input demand given p.*

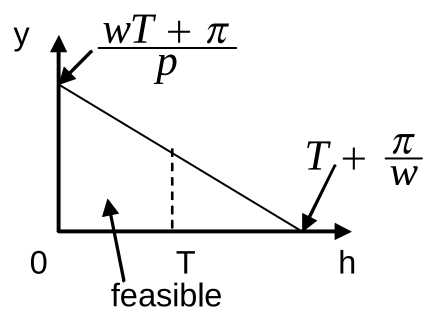
Theorem 9.6 *If $\succcurlyeq^i$ are continuous, strictly increasing and convex and if our assumptions on the production sets are satisfied then WE exists.*

9.1 Robinson Crusoe Economy

Suppose there is one consumer and one firm owned by the same consumer. We assume that consumption and production decisions are made independently. There are two goods, h time and y coconuts. The production set is

$$y = \{(-h, y), 0 \leqslant h \leqslant b, 0 \leqslant y \leqslant h^{\alpha}\}(0 < \alpha < 1)$$

The utility function is $u(h, y) = h^{1-\beta}y^{\beta}$, initial endowment is $e = (T, 0)$. p is price of coconuts, w is price of time.



The firm's problem is

$$\begin{cases} \max\limits_{h \geqslant 0} py - wh \\ y = h^{\alpha} \end{cases} \Rightarrow \max\limits_{h} ph^{\alpha} - wh$$

First order condition

$$\alpha p h^{\alpha - 1} = w \Rightarrow h_f = (\frac{w}{\alpha p})^{\frac{1}{\alpha-1}} = (\frac{\alpha p}{w})^{\frac{1}{1-\alpha}}, y_f = h_f^{\alpha} = (\frac{\alpha p}{w})^{\frac{\alpha}{1-\alpha}}.$$

Thus,

$$\pi = py_f - wh_f = \frac{1-\alpha}{\alpha} \cdot w \cdot (\frac{\alpha p}{w})^{\frac{1}{1-\alpha}}.$$

The consumer's problem is

$$\begin{cases} \max h^{1-\beta}y^{\beta} \\ st. py + wh = wT + \pi \end{cases}$$

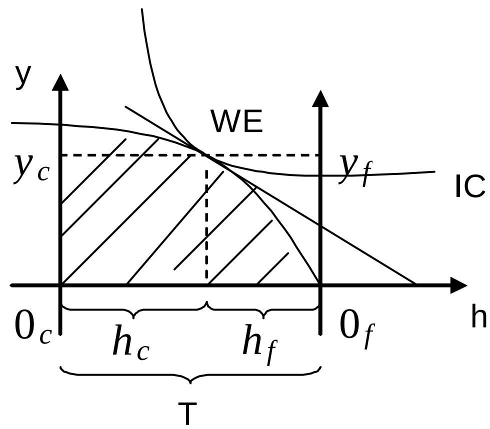
First order condition

$$\frac{y(1-\beta)}{h\beta} = \frac{w}{p} \Rightarrow y_c = \frac{\beta(wT+\pi)}{p}, h_c = \frac{(1-\beta)(wT+\pi)}{w}.$$

Suppose $p^* = 1$, then $h_c(1, w) + h_f(1, w) = T$, that is

$$\frac{(1-\beta)(wT+\pi)}{w} + (\frac{\alpha p}{w})^{\frac{1}{1-\alpha}} = T$$
$$\Rightarrow w^* = \alpha(\frac{1-\beta(1-\alpha)}{\alpha\beta T})^{1-\alpha}$$

From the above equations, we can see that as T increases, w^* decreases and as T decreases, w^* increases. Robinson Crusoe economy can be illustrated by the following picture, the shadow area is the feasible allocation of this economy. Robinson Crusoe cannot do any better than WE.



9.2 Welfare Theorems in Economies with Production

Consider an economy $(u^i, e^i, \theta^{ij}, y^j), i = 1..I; j = 1..J$.

Definition 9.7 *An allocation (x, y) is feasible if $x^i \in \mathbb{R}^n_+$, and $y^j \in Y^j, \forall i, j$, and*

$$\Sigma x^i = \Sigma e^i + \Sigma y^j.$$

Definition 9.8 *A feasible allocation is PE if there is no other feasible allocation $(\overline{x}, \overline{y})$ such that $\overline{x}^i \succcurlyeq^i x^i$, $\forall i$ with at least one being strict.*

In the definition we do not look at firms because they are owned by consumers.

Theorem 9.9 (First Welfare Theorem) *Any WE is PE.*

Proof. Suppose (x, y) is WE and not PE. Then (x, y) is feasible and $\sum_i x^i = \sum_i e^i + \sum_j y^j$. There is feasible allocation $(\overline{x}, \overline{y})$ st. $\overline{x}^i \succcurlyeq^i x^i$, $\forall i$ with at least one strict. Let p^* is equilibrium price, then$\overline{x}^k > x^k \Rightarrow p^*\overline{x}^k > p^*x^k$ and $\overline{x}^i \succcurlyeq^i x^i \Rightarrow p^*\overline{x}^i \geq p^*x^i$ for $i \neq k$. Consequently, $p^* \sum \overline{x}^i > p^* \sum x^i$, and so $p^*(\sum e^i + \sum \overline{y}^j) > p^*(\sum e^i + \sum y^j)$. Thus $p^*\overline{y}^j > p^*y^j$ and so $\exists j'$ $p^*\overline{y}^{j'} > p^*y^{j'}$. This means that firm j' did not maximize it's profit, which is contradiction. ■

Theorem 9.10 (Second Welfare Theorem) *If (x, y) is PE, then there is income re-distribution $T_1, , T_I$, such that $\Sigma T_i = 0$, and (x, y) is WE allocation of a new economy after transfers.*

Proof. No proof ■

9.3 Adding time and uncertainty to GE models

Example 9.11 *Assume there are contingent markets for two contingent goods x_1, x_2. For example, suppose there are two states: warm weather tomorrow and cold weather tomorrow, x_1 is ice-cream in warm weather(in state 1) and x_2 is ice-cream in cold weather (in state 2), when you buy one unit of x_1, you receive one unit of ice-cream only if the weather is warm.*

Suppose there are two consumers, two states of uncertainty and one good that will correspond to two contingent goods. The utility function is $u'(x_1, x_2) = q \ln x_1 + (1 - q) \ln x_2$, and endowments are $e^1 = (2, 1)$,

$e^2 = (1,2)$. *Consumer's demands are*

$$x_1^1(p_1,p_2) = \frac{q(2p_1+p_2)}{p_1},\ x_1^2(p_1,p_2) = \frac{q(p_1+2p_2)}{p_1}.$$
$$x_2^1(p_1,p_2) = \frac{(1-q)(2p_1+p_2)}{p_2},\ x_2^2(p_1,p_2) = \frac{(1-q)(p_1+2p_2)}{p_2}.$$

The market clears when $x_1^1(p_1,p_2) + x_1^2(p_1,p_2) = 3$. *Set* $p_1 = 1$ *and then*

$$\frac{q(2p_1+p_2)}{1} + \frac{q(p_1+2p_2)}{1} = 3$$
$$\Rightarrow p_2 = \frac{1-q}{q}$$
$$\Rightarrow x_1^1 = x_1^2 = 1+q, x_2^1 = x_2^2 = 2-q$$

This is WE allocation. The agents face risk before the trade, but after the trade, they consume the same amount of x_1 *and* x_2*−full insurance. Also notice that higher* q *implies lower* p_2. *This is intuitive because for higher* q *state 2 is less probable.*

Example 9.12 *Add time to GE model. Suppose there is one good* x *and two periods, that is* x_1 *is a consumption of* x *at period* 1 *and* x_2 *is a consumption of* x *at period* 2. *The utility function is* $u'(x_1,x_2) = \ln x_1 + \delta \ln x_2, \delta \in (0,1)$. δ *is time preference. Endowments are* $e^1 = (2,1), e^2 = (1,2)$. *Then WE allocation is*

$$x_1^1(p_1,p_2) = \frac{1}{1+\delta}(2+\frac{p_2}{p_1}),$$
$$x_1^2(p_1,p_2) = \frac{1}{1+\delta}(1+\frac{2p_2}{p_1}).$$

Market clears when $x_1^1(p_1,p_2) + x_1^2(p_1,p_2) = 3$. *Let* $p_1 = 1$, *then* $p_2 = \delta, x_1^1 = x_2^1 = 1 + \frac{1}{1+\delta}$ *and* $x_1^2 = x_2^2 = 2 - \frac{1}{1+\delta}$.

10 Externalities

Why market failures exist in competitive market? There are three reasons: externalities, monopoly and incomplete information.

Definition 10.1 *We say that there is an externality when one agent is directly affected by actions of another agent, such as loud music, pollution, tidy roommate and so on.*

Example 10.2 *Assume there are two consumers and two goods. Initial endowment are* $e^1 = (4,0), e^2 = (6,4)$. *Utility function for the two agents are* $u^2(x_1^2,x_2^2) = x_1^2 + 2\sqrt{x_2^2}, u^1(x_1^1,x_2^1,x_2^2) = x_1^1 + 2\sqrt{x_2^1} - \sqrt{x_2^2}$. *Let* $p_1 = 1$, *then for the first agent:*

$$\begin{cases} \max_{x_1,x_2} x_1^1 + 2\sqrt{x_2^1} - \sqrt{x_2^2} \\ st. x_1^1 + p_2 x_2^1 = 4 \end{cases}$$

This is equivalent to

$$\max_{x_2^1} 4 - p_2 x_2^1 + 2\sqrt{x_2^1} - \sqrt{x_2^2}.$$

First order condition

$$-p_2 + \frac{1}{\sqrt{x_2^1}} = 0 \Rightarrow x_2^1 = (\frac{1}{p_2})^2.$$

For the second agent,

$$\begin{cases} \max x_1^2 + 2\sqrt{x_2^2} \\ st.x_1^2 + p_2 x_2^2 = 6 + 4p_2 \end{cases}$$

This is equivalent to

$$\max 6 + 4p_2 - p_2 x_2^2 + 2\sqrt{x_2^2}.$$

First order condition

$$-p_2 + \frac{1}{\sqrt{x_2^2}} = 0 \Rightarrow x_2^2 = (\frac{1}{p_2})^2.$$

Market clears when $(\frac{1}{p_2})^2 + (\frac{1}{p_2})^2 = 4 \Rightarrow p_2 = \frac{1}{\sqrt{2}}$, *then the WE allocation is* $x^1 = (4-\sqrt{2}, 2), x^2 = (6+\sqrt{2}, 2)$. *Now we check whether WE is PE*

$$MRS^1 = \frac{MU_1^1}{MU_2^1} = \frac{MU_1^2}{MU_2^2} = MRS^2$$

$$MRS^1 = \frac{1}{\frac{1}{\sqrt{x_2^1}} + \frac{1}{2\sqrt{4-x_2^1}}} = MRS^2 = \frac{1}{\frac{1}{\sqrt{x_2^2}}} = \frac{1}{\frac{1}{2\sqrt{4-x_2^1}}}$$

$$\Rightarrow x_2^1 = \frac{16}{5}, x_2^2 = \frac{4}{5}$$

This is PE allocation. With externalities, WE is not PE, because the second agent eats too much of good two. When the second agent maximizes his utility he ignores the adverse effect on the first agent and this is why he imposes too much of a negative externality on agent one.

10.1 Remedies

There are three major remedies for externalities: quotas, taxes/subsidies and property rights assignment.

1. **Quotas**: agent two can consume at most $\frac{4}{5}$ of good two. Assume $p_1 = 1, p_2 = \sqrt{\frac{5}{16}}$, then $x_2^1 = \frac{16}{5}$. The demand of $x_2^2 = 2$, but given our restriction agent two will eat only $\frac{4}{5}$.

2. **Taxation. Pigouvian taxes**. The second agent has to pay tax t on each unit of x_2 he consumes. In addition the second agent receives (pays) a transfer of T_2 so that his maximization problem becomes

$$\begin{cases} \max x_1^2 + 2\sqrt{x_2^2} \\ st.p_1 x_1^2 + (p_2 + t)x_2^2 = 6p_1 + 4p_2 + T_2 \end{cases}$$

The maximization problem for the first agent is

$$\begin{cases} \max_{x_1, x_2} x_1^1 + 2\sqrt{x_2^1} - \sqrt{x_2^2} \\ st.p_1 x_1^1 + p_2 x_2^1 = 4p_1 + T_1 \end{cases}$$

In addition we require that $T_1 + T_2 = t x_2^2$. We would like to find t that would make agent 2 to consume optimal amount of good 2. From the FOCs for agent 2 we get that $x_2^2 = \frac{1}{(p_2)^2}$ and so the equilibrium price should be equal to $p_2 = \sqrt{\frac{5}{16}}$. When prices are $p_1 = 1, p_2 = \sqrt{\frac{5}{16}}$ we have that

$$FOC : \frac{1}{(p_2 + t)^2} = x_2^2 = \frac{1}{(\sqrt{\frac{5}{16}} + t)^2} \Rightarrow t = \frac{\sqrt{5}}{4}$$

We can check that the market clears regardless of T_1 and T_2.

2'. **Subsidy** to consumer two for each unit below 4. The budget constraint for consumer two is

$$p_1 x_1^2 + p_2 x_2^2 = 6p_1 + 4p_2 + s(4 - x_2^2) - \text{taxes}$$

The government needs to have money to subside consumer 2. This is why it will tax somehow both consumers. The budget constraint is equivalent to

$$p_1 x_1^2 + (p_2 + s)x_2^2 = 6p_1 + 4p_2 + 4s - \text{taxes}$$

Thus we see that subsidy is the same as taxes.

3. **Assigning Property Rights:** Person one has a right to externality-free environment, that is agent one has right to make $x_2^2 = 0$. Agent two has initial endowment $(6, 4)$, but can consume only $(6, 0)$. The idea is that agent two can pay to agent one T and give to agent one $4 - x_2^2$ units of good 2 for the right to consume x_2^2. For the first agent,

$$\max_{T, x_2^2} 4 + T + 2\sqrt{4 - x_2^2} - \sqrt{x_2^2}$$

$$st.\ 6 - T + 2\sqrt{x_2^2} \geqslant 6$$

In optimum the constraint will hold with equality and then $6 - T + 2\sqrt{x_2^2} = 6 \Rightarrow T = 2\sqrt{x_2^2}$, so

$$\max_{T, x_2^2} 4 + 2\sqrt{x_2^2} + 2\sqrt{4 - x_2^2} - \sqrt{x_2^2}$$

$$FOC: \frac{1}{2\sqrt{x_2^2}} = \frac{1}{\sqrt{4 - x_2^2}}$$

$$\Rightarrow x_2^2 = \frac{4}{5}$$

PE is restored here. The first agent can negotiate with the second agent before trade.

Alternatively, we could give all rights to consumer 2, that is he would have the right to produce as much externalities as he wants. He can make take-it or leave-it offer to agent one, that is "*give me T of good one, I reduce my consumption of good 2 to x_2^2 and you can get the rest of x_2.*" For the second agent,

$$\max_{T, x_2} 6 + T + 2\sqrt{x_2^2}$$

$$st. 4 - T + 2\sqrt{4 - x_2^2} - \sqrt{x_2^2} \geqslant 4 - \sqrt{4}$$

It's optimal to have " $=$ ", then we can get $4 - T + 2\sqrt{4 - x_2^2} - \sqrt{x_2^2} = 4 - \sqrt{4} \Rightarrow T = \sqrt{4} + 2\sqrt{4 - x_2^2} - \sqrt{x_2^2}$, so

$$\max\ 6 + \sqrt{4} + 2\sqrt{4 - x_2^2} + \sqrt{x_2^2}$$

$$FOC: \frac{1}{2\sqrt{x_2^2}} = \frac{1}{\sqrt{1 - x_2^2}}$$

$$\Rightarrow x_2^2 = \frac{4}{5}$$

In other words no matter how you assign property right, it is possible to restore efficiency.

Theorem 10.3 (Coase Theorem) *If property rights for externality are assigned, and negotiating is costless, then it will result in an efficient outcome, no matter how property rights are assigned.*

10.2 Public goods

A public good is a commodity for which the use of a unit by one agent does not preclude its use by other agents. There are two kinds of public good, the one is excludable public good, when it is possible to exclude somebody to consume it, such as patent system, knowledge and toll roads; the other one is non-excludable public good, such as national defense. The problem we face with public goods is that consumers don't take into account positive effect of PG on other people, which leads to underprovision and free riding.

Example 10.4 *Assume there are two consumers, one firm and two goods: time and a public good. Initial endowment is $e^i = (T, 0)$, the utility function is $u^i(x^i, x^j, l^i) = 2\sqrt{x^i + x^j} + l^i$, where $x^i + x^j$ is the total amount of PG. The firm uses 2 units of labor to produce 1 unit of PG, that is $y = \frac{1}{2}l$. Let $w = 1$ then consumer's maximization problem is*

For consumers

$$\begin{cases} \max 2\sqrt{x^i + x^j} + l^i \\ st.px^i = (T - l^i) \cdot 1 \end{cases} \Rightarrow \max 2\sqrt{x^i + x^j} + T - px^i$$

$$x^i = \frac{1}{p^2} - x^j, x^j = \frac{1}{p^2} - x^i$$

For firms,

$$\max p \cdot \frac{1}{2}l - l$$

$$\textit{if } p = 2 \Rightarrow l = [0, +\infty)$$

$$\textit{if } p < 2 \Rightarrow l = 0$$

The second case is impossible in equilibrium, so it should be that $p = 2$, and thus $x^i + x^j = \frac{1}{p^2} = \frac{1}{4} > 0$. In a symmetric equilibrium, $x^1 = x^2 = \frac{1}{8}$, $l^1 = l^2 = T - \frac{1}{4}$, consumer's utility $u^1 = 2\frac{1}{\sqrt{4}} + T - \frac{1}{4} = T + \frac{3}{4} = u^2$.

Now let's look at the symmetric efficient allocation. Assume each consumer works h hours, $u^i = 2\sqrt{h} + T - h$, $h = \frac{h_1 + h_2}{2}$.

$$\max_h u^i = 2\sqrt{h} + T - h$$

$$\frac{1}{\sqrt{h}} = 1 \Rightarrow h = 1$$

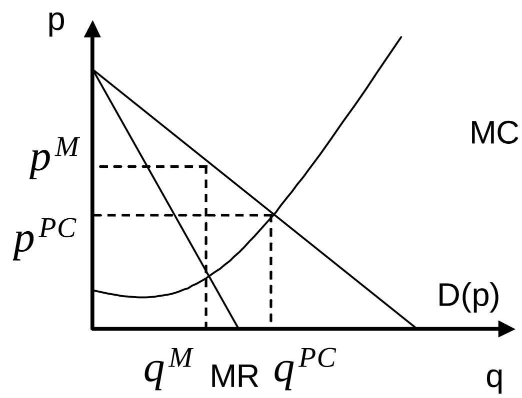
$$u^i = 2 + T - 1 = T + 1 > T + \frac{3}{4}$$

In PE allocation, utilities are higher and consumers work more than in WE allocation.

11 Monopoly

We are now going back to the partial equilibrium framework. However, now we assume now that there is only one firm. It maximizes its profit given its technology (just as before) and consumer behavior in other words given the consumer's demand. The maximization problem is

$$\begin{cases} \max_q \; pq - C(q) \\ st.p = p(q) \end{cases} \Rightarrow \max \; p(q) \cdot q - C(q)$$



First order condition $p'(q) \cdot q + p(q) - C'(q) = 0$ and so $MR = MC$, where $MR = p'q + p$. The intuition is that as you change q there are two effects: first as $q \to q + \Delta q$ the revenue goes up by $p(q)\Delta q$ and at the same time price falls down by $p'(q)\Delta q$. The firm looses on each unit it sells so revenue decreases by $p'(q)q\Delta q$. Total effect is $R'(q) = p(q) + p'(q)q$. Notice also that since $p'(q) < 0$ it follows that $R'(q) < p(q)$.

Consider the price elasticity of demand $\varepsilon(q) = q'(p) \cdot \frac{p}{q}$.

$$MR = p'(q) \cdot q + p(q) = p(q)[1 + p'(q)\frac{q}{p(q)}]$$
$$p'(q) = \frac{1}{q'(p)} = p(q)[1 + \frac{1}{\varepsilon(q)}]$$
$$MR = MC \Rightarrow p(q)[1 + \frac{1}{\varepsilon(q)}] = MC$$

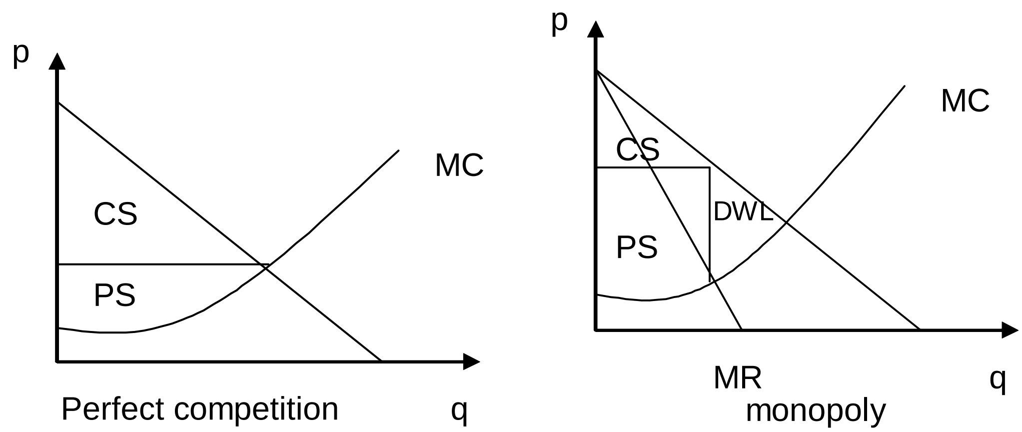
When $|\varepsilon(q)| < 1$ the demand is inelastic; $p(q)[1 + \frac{1}{\varepsilon(q)}] < 0$, which cannot be equal to MC, so monopoly will always choose ∞ price. If $|\varepsilon(q)| > 1$, then the demand is elastic and the price will be determined from $MC = p(q)[1 + \frac{1}{\varepsilon(q)}]$.And mark-up is $\frac{p - MC}{p} = -\frac{1}{\varepsilon(q)}$, that is the higher is absolute value of demand elasticity the smaller is the mark-up.

Example 11.1 *Assume a constant elasticity utility function $q(p) = A \cdot p^{-b}$, then*

$$q'(p) = -bA \cdot p^{-b-1}$$
$$\varepsilon(q) = -bA \cdot p^{-b-1} \cdot \frac{p}{A \cdot p^{-b}} = -b$$
$$MC = p(1 - \frac{1}{b}) = c \Rightarrow p = \frac{cb}{b-1}$$

We can see that the higher is b the less is p, and $|\varepsilon(q)| = b$ has to be greater than 1, or the monopoly price will be ∞.

The pictures below show the comparison of consumer surplus and producer surplus in a perfectly competitive market and in a monopolized market.



12 Price discrimination

Definition 12.1 *Price discrimination is selling different units of the same goods for different price, either to the same or to different consumers.*

The most common examples of price discrimination are student discounted tickets and wholesale discounts.

There are three kinds of price discrimination.

1. First-degree PD: sellers charges different prices for each unit of good in such a way that the charged price is equal to consumer's maximum willingness to pay for this unit.

2. Second-degree PD: price depends only on quantity purchased, but is the same across consumers.

3. Third-degree PD: different consumers pay different prices, but each consumers pay the same price for each unit, such as student discount.

Assume there are two consumers with utility $u_i(x) + m$. We can think of $u_i(x)$ as the utility from consuming the good and m as the utility of money. If $u_i(0) = 0$, then what is the maximum WTP for x units of good?

$$u(x, m) = u_i(x) - r_i(x) \geqslant u_i(0)$$
$$\Rightarrow r_i(x) \leqslant u_i(x)$$

Max WTP is exactly $u_i(x)$. If we put price p into this problem, then

$$\max_x u_i(x) - px.$$

By first order condition, we get consumer inverse demand function $u_i'(x) = p$.

Now we can proceed to analyzing each type of price discrimination. In the analysis we will assume that $u_2(x) > u_1(x)$, $u_2'(x) > u_1'(x)$ and $c(x) = cx$.

12.1 First-degree PD

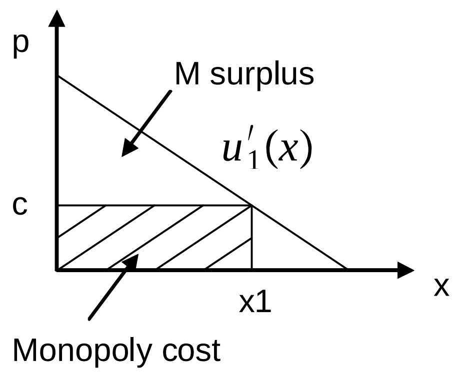
Monopoly observes $u_i(x)$ and makes take-it or leave-it offer to each consumer. Consumer i can buy x_i units of the good and has to pay $u_i(x_i)$. The monopoly's problem is

$$\max_{x_1, x_2} u_1(x_1) - cx_1 + u_2(x_2) - cx_2$$

First order condition

$$u_1'(x_1) = c, u_2'(x_2) = c$$

Here is the picture illustrating first-degree PD. x_1 is the same amount that the consumer would receive in a perfectly competitive market. However, under perfect competition, the consumer would pay cx_1, and in the case of monopoly, the consumer pays $u_1(x_1) = \int\limits_0^{x_1} u_1'(x_1)dx$, which is the sum of monopoly surplus and monopoly cost.



In first-degree PD, the allocation coincides with the competitive allocation and thus it is efficient (max total surplus). However, it is the monopoly who gets this whole surplus. In contrast, in PC market, it were consumers who would get all the surplus.

12.2 Second-degree PD

There is one good, one producer (monopolist) are two consumers 1 and 2, both have utility function $u_i(x) - m$. That is if they consumer x units of the good and pay m dollars their utility is $u_i(x) - m$. We assume that the first consumer values the good less than the second that is $u_1(x) < u_2(x)$, and moreover that $u_1'(x) < u_2'(x)$.

The monopolist has a fixed marginal cost c of providing one unit of good. He wants to offer two bundles (x_1, r_1) and (x_2, r_2) to these two consumers to maximize his revenue. Since the monopolist does not know the type of consumer he needs to make sure that the first consumer prefers to buy the first bundle. Thus, his maximization problem is

$$\begin{cases} \max_{r_1, x_1, r_2, x_2} r_1 - cx_1 + r_2 - cx_2 & \\ u_1(x_1) - r_1 \geq 0 & (IR_1) \\ u_2(x_2) - r_2 \geq 0 & (IR_2) \\ u_1(x_1) - r_1 \geq u_1(x_2) - r_2 & (IC_1) \\ u_2(x_2) - r_2 \geq u_2(x_1) - r_1 & (IC_2) \end{cases}$$

Solution. The first step is to show that (IR_1) is binding and (IR_2) is not. We start with the latter by showing that (IR_2) follows from (IR_1) and (IC_2). Indeed,

$$u_2(x_2) - r_2 \geq u_2(x_1) - r_1 \geq u_1(x_1) - r_1 \geq 0.$$

The first inequality is (IC_2), the second is assumption that $u_1(x) \leq u_2(x)$ and the third is (IR_1).

Now we can show that (IR_1) should be satisfied with equality in the optimum. Assume not. That is assume that (x_1, r_1) and (x_2, r_2) solve and $u_1(x_1) - r_1 > 0$. Then consider bundles $(x_1, r_1 + \varepsilon)$ and $(x_2, r_2 + \varepsilon)$. These two new bundles will give the monopolist higher profit. Moreover since the original bundles satisfied (IC_1) and (IC_2) the new bundles will satisfy them too. Finally as long ε is small enough, individual rationality constraints will be also satisfied. Our problem now becomes

$$\begin{cases} \max_{r_1,x_1,r_2,x_2} r_1 - cx_1 + r_2 - cx_2 & \\ u_1(x_1) - r_1 = 0 & (IR_1) \\ u_1(x_1) - r_1 \geq u_1(x_2) - r_2 & (IC_1) \\ u_2(x_2) - r_2 \geq u_2(x_1) - r_1 & (IC_2) \\ \end{cases}$$

Let us solve it without (IC_1). Then we will show that the solution of the reduced problem will satisfy (IC_1) and thus will be the solution to the original problem. The reduced problem is

$$\begin{cases} \max_{r_1,x_1,r_2,x_2} r_1 - cx_1 + r_2 - cx_2 & \\ u_1(x_1) - r_1 = 0 & (IR_1) \\ u_2(x_2) - r_2 \geq u_2(x_1) - r_1 & (IC_2) \\ \end{cases}$$

We can immediately see that (IC_2) should hold with equality because otherwise we can just increase r_2. Thus the problem becomes

$$\max_{x_1,x_2}(u_1(x_1) - cx_1) + (u_2(x_2) - u_2(x_1) + u_1(x_1) - cx_2)$$

First-order conditions are $u_2'(x_2) = c, \quad u_1'(x_1) = c + u_2'(x_1) - u_1'(x_1) > c$.The immediate conclusion is that the second person consumes optimal amount of good and the first person consumes less than optimal amount of good. Now we need to justify that (IC_1) is satisfied by the solution that we found. First we show that whenever $x_1 \leq x_2$ then (IC_1) will follow from (IR_1) and (IC_2). From (IR_1) and (IC_2) we get

$$r_2 = u_2(x_2) - u_2(x_1) + u_1(x_1).$$

Given that (IR_1) is satisfied with equality, (IC_1) becomes

$$0 \geq u_1(x_2) - r_2.$$

Plugging r_2, we have that we need to show that

$$0 \geq u_1(x_2) - u_2(x_2) + u_2(x_1) - u_1(x_1).$$

But indeed we can re-write it as

$$u_2(x_2) - u_2(x_1) \geq u_1(x_2) - u_1(x_1) \Leftrightarrow \int_{x_1}^{x_2} u_2'(x)dx \geq \int_{x_1}^{x_2} u_1'(x)dx.$$

We assumed that $u_2'(x) \geq u_1'(x)$ and thus as long as $x_2 \geq x_1$ we have that (IC_1) is satisfied.

Thus to show that (IC_1) is satisfied by the solution that we found we only need to show that $x_2 \geq x_1$. It makes our life easier because using only it would be hard to verify it.

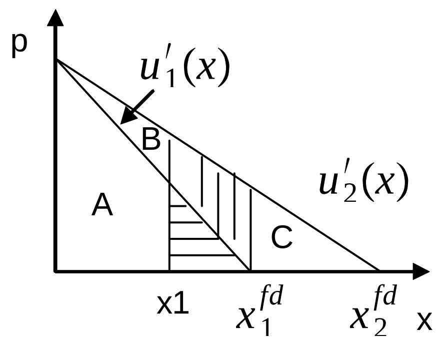
Let us check that in the solution that we found $x_2 \geq x_1$. Indeed,

$$u_2'(x_1) > u_1'(x_1) > c \qquad u_2'(x_2) = c.$$

The first inequality is our assumption, the *second* one follows from the FOC. Given that $u_i(x), i = 1, 2$ are concave function we have that $x_2 \geq x_1$.

The main interpretation is that the monopolist extract the whole surplus from the first agent (who values the good less), and the second agent enjoys some positive surplus. The second agent also consumes efficient level of good, whereas the first person consumes less than $efficient$ level. ■

Now we illustrate the second-degree PD. Assume $c = 0$, if monopoly knows who is who and asks x_1^{fd} pay $A = \int_0^{x_1} u_1'(x_1)dx = u_1(x_1)$, x_2^{fd} pay $A + B + C = \int_0^{x_1} u_2'(x_2)dx = u_2(x_2)$. But if the monopoly doesn't know who is who, then the second agent gets 0 or B. So how monopoly can induce the second agent to buy x_2^{fd}? Monopoly has to provide (x_1^{fd}, A) and provide $(x_2^{fd}, A+C)$, then the first agent will buy (x_1^{fd}, A), the second agent is indifferent to buy $(x_2^{fd}, A + C)$, because the surplus of the second agent is B for both.



As the picture shows, if the monopoly provides $(x_1^{fd}, A-$shadowed triangle), surplus of the second agent is $B-$shadowed ladder area); if the monopoly provides $(x_2^{fd}, A + C+$shadowed ladder area), surplus of the second agent is $B-$shadowed ladder area). So the second agent is indifferent. Although monopoly loses the shadowed triangle on the first agent, but gains the shadowed ladder area on the second agent, so we will move x_1 to the left until shadowed triangle is equal to shadowed ladder area.

12.3 Third-degree PD

Assume there are two markets and two different demand functions.

$$q_1 = p_1(x), q_2 = p_2(x)$$

The monopoly's problem is

$$\max_{x_1, x_2} p_1(x_1)x_1 - cx_1 + p_2(x_2)x_2 - cx_2$$

First order condition

$$\begin{aligned} p_1(x_1) + p_1'(x_1)x_1 &= c \\ p_2(x_2) + p_2'(x_2)x_2 &= c \end{aligned}$$

This is equivalent to

$$\begin{aligned} p_1(x)(1 - \frac{1}{|\varepsilon|}) &= c \\ p_2(x)(1 - \frac{1}{|\varepsilon|}) &= c \end{aligned}$$

So the monopolist charges a lower price on a market with more elastic demand.

13 Social choice and welfare

Society needs to choose from several different alternatives, such as political candidates or "how to divide a pie". However, given that different members have different preferences how the society should decide?

Let X denote the set of all possible alternatives, $N \geq 2$ denotes a number of individuals and R^i denotes preferences of individual i so that

$$xR^iy \Leftrightarrow x \succcurlyeq^i y \text{ weak preference}$$
$$xP^iy \Leftrightarrow x \succ^i y \text{ strict preference}$$
$$xI^iy \Leftrightarrow x \sim^i y \text{ indifference}$$

R^i is preference relation, which means it's complete and transitive. Each person i can rank all available allocations $(x_1, x_2, .., x_z)$ according to R^i. We want to derive social choice that represents the preference in the society. Define R as social ranking, "xRy", means the society prefers x to y, and we want R to be complete and transitive. Similarly, let P denote social strict preference and I to denote social indifference.

Example 13.1 *Majority Rule: assume xRy, iff xR^iy by majority of people, that is there are at least $\frac{N}{2}$ of those who prefers x to y.The majority rule is complete but not transitive.*

Condorcet Paradox*: There are three persons and three options (x, y, z). The first one ranks as (x, y, z), the second one ranks as (y, z, x), the third one ranks as (z, x, y). Based on the majority rule the society would ranks the three options as xRy, yRx, zRx, which is a contradiction (because it's not transitive). Thus if we want R to be transitive, we cannot use majority rule.*

Our next step is to try to find out such a social choice function that takes a vector of individual preferences $R^1, R^2, , R^N$ and returns society preferences R. For example,

$$f(\left\{\begin{array}{c} x \\ y \\ z \end{array}\right\}, \left\{\begin{array}{c} y \\ z \\ x \end{array}\right\}, \left\{\begin{array}{c} z \\ x \\ y \end{array}\right\}) = (\left\{\begin{array}{c} x \\ y \\ z \end{array}\right\})$$

Requirements:

1. **U** (universal domain). Domain of f must be any combination of preferences over x.

2. **WP** (weak Pareto principle). $\forall x, y$ if xP^iy, then $\forall i\ xPy$.

3. **IIA** (independence from irrelevant alternatives). Define $R = f(R^1, R^2, , R^N)$, $\widetilde{R} = f(\widetilde{R}^1, \widetilde{R}^2, , \widetilde{R}^N)$, if each person i ranks x over y under R^i in the same way as under $\widetilde{R}^i$,then xRy iff $x\widetilde{R}y$. Social ranking of x and y does not depend on other alternatives.

4. **D** (non-dictatorship). There is no such $i, \forall x, y\ xRy$ iff xR^iy regardless of preference of other people.

Theorem 13.2 (Arrow theorem) *If there are at least three alternatives in X, then the only social choice function f that satisfying U, WP, IIA is a dictator social choice function. Note that if $|X| = 2$, then the majority rule works.*

Another name is Arrow Impossible theorem, which says that there is no function that satisfying U, WP, IIA, D.

Proof. *Step 1*: Take an element $c \in X$ and consider a vector of preferences such that each i puts c in the bottom then the society should put c on the bottom as well.

$$\begin{array}{ccccccc} R_1 & R_2 & & R_n & & R_N & & R \\ \left\{\begin{array}{c} x \\ y \\ .. \\ c \end{array}\right\} & \left\{\begin{array}{c} x' \\ y' \\ .. \\ c \end{array}\right\} & \ldots & \left\{\begin{array}{c} x'' \\ y'' \\ .. \\ c \end{array}\right\} & \ldots & \left\{\begin{array}{c} x''' \\ y''' \\ .. \\ c \end{array}\right\} & \Rightarrow & \left\{\begin{array}{c} .. \\ .. \\ .. \\ c \end{array}\right\} \end{array}$$

Step 2: Take person 1 and lift c from the bottom to top, and do it for everyone. During the process, there will be the first time that the social ranking of c increases. Let n be the first individual when it happens.

$$\begin{array}{ccccccc} R_1 & R_2 & & R_n & & R_N & & R \\ (*)\left\{\begin{array}{c} c \\ x \\ y \\ .. \\ .. \end{array}\right\} & \left\{\begin{array}{c} c \\ x' \\ y' \\ .. \\ .. \end{array}\right\} & \ldots & \left\{\begin{array}{c} c \\ x'' \\ y'' \\ .. \\ .. \end{array}\right\} & \ldots & \left\{\begin{array}{c} x''' \\ y''' \\ .. \\ .. \\ c \end{array}\right\} & \Rightarrow & \left\{\begin{array}{c} c \\ .. \\ .. \\ .. \\ .. \end{array}\right\} \end{array}$$

Claim: when c comes to the top of R_n, the social ranking of c not just moves up, but moves up to the top. Now we prove this claim.

Assume $\exists a, b$ $aRcRb$, $a \neq c, b \neq c$. Change individual preferences so that $bP^i a$ for each i, while keeping the position of c unchanged. By WP, we get bPa. At the same time since ranking between a and c and between b and c did not change for any i. Thus we can use IIA to conclude that aRc and cRb, then by transitivity aRb, which contradicts with bPa. So c should be on the top of societal choice.

Step 3: Take $a \neq b \neq c$. Preferences are the same as in (*). Now change the preference of person n, so that $aP^n cP^n b$. For everyone else, change preferences of a and b in any way without changing the position of c. By IIA, the individual ranking between a and c is the same as one step before (*), thus we get aPc. By IIA, the individual ranking between b and c is the same as on (*), thus we get cPb. By transitivity, we can get aPb.

It shows that no matter how person $j \neq n$, ranks a and b, aPb whenever $aP^n b$. In other words, social ranking of a and b is consistent with n's ranking for any $a, b, a \neq c, b \neq c$. So n is the dictator for all pairs that do not involve c.

Step 4: Now we want to show that n is dictator. Take another element of x, say $d \neq c$ and repeat all the steps above. We can find the dictator for all pairs that do not involve d. However, the way n ranks c affects social ranking. Thus it has to be that n is the dictator. ■

One of the way to get around the Arrow theorem is to relax one of the assumptions. For example we can relax U since everyone prefers more to less. Another option is to relax IIA.

Example 13.3 *Assume there are two people with preferences. Let*

$$\begin{array}{cc} 1 & 2 \\ \left\{\begin{array}{c} x \\ y \\ .. \end{array}\right\} & \left\{\begin{array}{c} y \\ .. \\ .. \end{array}\right\} \end{array}$$

and yRx. Change preferences of agent 1 in such a way that

$$\begin{array}{cc} 1 & 2 \\ \left\{ \begin{array}{c} x \\ .. \\ y \end{array} \right\} & \left\{ \begin{array}{c} y \\ x \\ .. \end{array} \right\} \end{array}$$

then by IIA it still has to be the case that yRx. However, consider a social choice function that works as follows. It assigns 10 to the highest alternative, 9 to the second highest and so on with 0 to the lowest. Then these values are sum up across different agents and the alternative with the highest number wins. In this case for the original preferences we would get yRx and xRy for new preferences.

What happens in the example above is we start making interpersonal comparisons. We notice that in the first case the first agent does not dislike y too much, whereas in the second case he seriously dislikes y. We take it into account by not choosing y. More generally when we relax IIA and allow for interpersonal comparisons we can have social welfare functions which satisfies fairness or equity:

$$V(x) = \min\{u^1(x), \ldots, u^n(x)\}$$
$$V(x) = \sum_i u^i(x).$$

14 Information economics. Adverse Selection Models

One of the fundamental results in economics are welfare theorems that claim that under certain assumptions the market equilibrium is Pareto efficient outcome and in this sense is optimal. One of the assumptions needed for this result to hold is symmetric information which clearly does not hold in real life. For example, you as a buyer might have inferior knowledge about the product quality as compared to the seller. A simple illustration to that is a classical model of lemon market.

14.1 Simple adverse selection model. Lemon Market.

There are two types of cars: high- and low-quality cars. Buyers value high-quality cars as v_h and low-quality as v_l. We assume that $v_h > v_l$ and that $c_h > c_l$ that is it is costlier to produce a good car. We also assume that

$$v_h > c_h > c_l > v_l,$$

that is it is efficient to sell high-quality cars and inefficient cars should not be even produced.

Let α be a share of h-cars. We assume that

$$c_h > (1-\alpha)v_l + \alpha v_h > (1-\alpha)c_l + \alpha c_h.$$

The latter inequality means that there is ex-ante gain from trade. The former means that when buyers are uncertain about the car quality it is not profitable for h-sellers to sell their cars.

Proposition 14.1 *Under perfect information only high-quality cars are sold.*

Proof. This statement is so obvious I can't believe you are actually reading the proof. It is absolutely straightforward and immediately follows from the assumptions that we made. So stop reading. There. ■

Proposition 14.2 *When buyers cannot observe car's quality the equilibrium does not exist. Put it differently there is no price that will clear the market.*

Proof. Consider three cases:

- $p < c_l$: supply is zero and demand is positive;
- $c_l \leq p < c_h$: supply is $1 - \alpha$ and demand is zero;
- $p \geq c_h$: supply is 1 but demand is zero.

Thus there is no price when supply of cars is equal to the demand. ■

The intuition here is as follows. Normally when market does not clear the price is used to equilibrate it. For example, if supply is greater than demand the price will fall down and it will supply and demand meet. Here, it is not quite the case. When $p \geq c_h$ the supply is higher than demand. *As the price goes down the supply indeed decrease, however, so does the average car quality which prevents D from increasing.* What we have here is called *adverse selection* because it is good cars that are filtered out the first, and it is bad cars that stay on the market longest.

Assume there is an auto insurance market and two types of drivers: type L, the low risk driver, with a probability of being in an accident π_L; and type h, the high risk driver, with a probability of being in an accident π_h. We assume that $\pi_h > \pi_L$. We also assume that if accident happens, then the loss is the 1 regardless of the driver's type. Insurance company sells insurance, 1 units of insurance pays back \$$L$, thus its profit is $(p - \pi L)y$, in which p is the price of one unit of insurance, y is how much of insurance was purchased and finally π is the probability of the accident.

14.2 Insurance. Symmetric information

Insurance company knows which type of driver it deals with and sells two goods: insurance that pays L to type L, and insurance that pays L to type H. The maximization problem of consumer of type h if he buys insurance is

$$\max_{\theta} \pi_h u(w - L - p_h\theta + \theta L) + (1 - \pi_h)u(w - p_h\theta)$$
$$\pi_h(L - p_h)u'(w - L - p_h\theta + \theta L) - (1 - \pi_h)p_h u(w - p_h\theta) = 0$$

If $p_h = \pi_h L$, then we get $\theta = 1$, which means the risk aversion agent buys full insurance.

As for the insurance company, they want to max profit

$$\max_{y}(p - \pi_h L)$$

$$\text{if } p > \pi_h L \Rightarrow y = \infty$$
$$p < \pi_h L \Rightarrow y = 0$$
$$p = \pi_h L \Rightarrow y \in [0; \infty]$$

Thus in the equilibrium $p = \pi_h L$ and the high-type consumer buys one unit of insurance. Similarly, if $p_L = \pi_L L$, the low-type driver also buys one unit of insurance. In conclusion, in the equilibrium, price is fair and firms earn zero profit while consumers buy full insurance.

14.3 Insurance. Asymmetric information

Consumers know their type and firms do not know. Assume probability of accidents is given. Then the problem is similar to the second-degree price discrimination.

Suppose consumers can buy 0 or 1 unit of insurance at price p. The consumer will buy 1 unit of insurance if

$$\pi u(w-L)+(1-\pi)u(w)\leqslant u(w-p)$$

By transfers, we get

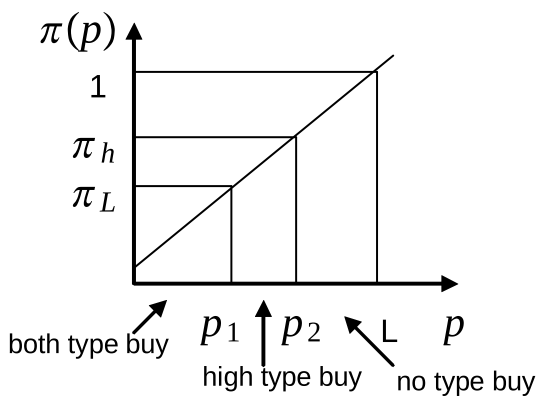
$$\pi\geqslant\frac{u(w)-u(w-p)}{u(w)-u(w-L)}=:g(p)$$

Properties of g:

1. $g(0)=0$.
2. $g(1)=1$.
3. $g(p)$ is an increasing function.

If $g(p)$ is the probability of an accident, then an agent would be indifferent between paying p for insurance or not. Agent observes p and if $\pi>g(p)$, he buys; if $\pi<g(p)$, he does not buy; if $\pi=g(p)$, he is indifferent.

Assume there are two price p_1,p_2, and $\pi_L=g(p_1),\pi_h=g(p_2)$. When $p_1>\pi_L L$, if the price is fair, the consumer will buy one unit. This can be illustrated by the following picture.



Now assume the probability of low-type is α, high-type is $1-\alpha$. If both types buy, then the expect profit is

$$[\alpha(p-\pi_L L)+(1-\alpha)(p-\pi_h L)]y$$

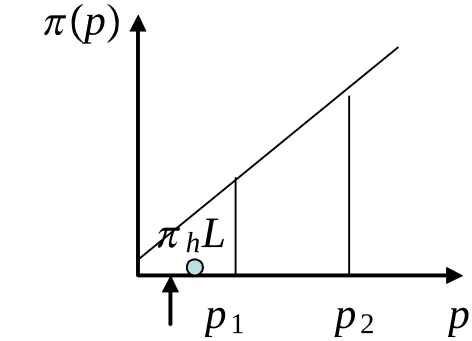
Just as before in equilibrium

$$\alpha(p-\pi_L L)+(1-\alpha)(p-\pi_h L)=0$$
$$p_{eq}=\alpha\pi_L L+(1-\alpha)\pi_h L$$

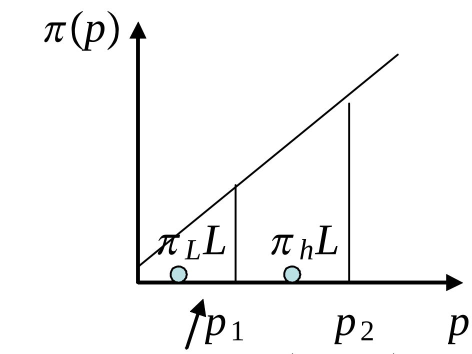
There are three insurance purchase conditions:

1. If $\pi_h L<p_1$, then there is only one equilibrium with $p=\alpha\pi_L L+(1-\alpha)\pi_h L$ and both types buy. Here $p=\pi_h L$ is not an equilibrium because both types buy insurance, then profit is positive, supply is infinite. This is efficient equilibrium because both types are insured.

2. If $\alpha\pi_L L+(1-\alpha)\pi_h L<p_1<\pi_h L$, then there are two equilibria with $p=\alpha\pi_L L+(1-\alpha)\pi_h L$ and both types buy and firms earn zero profit. When $p=\pi_h L$, only h type buys and firms earn zero profit.



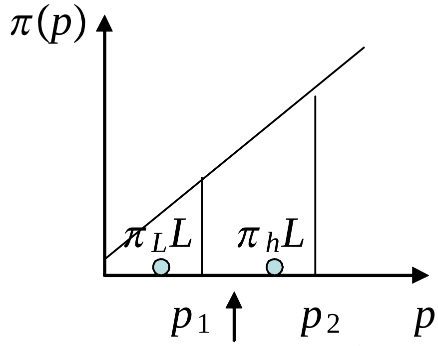
$$p_{eq} = \alpha\pi_L L + (1-\alpha)\pi_h L$$



$$p_{eq} = \alpha\pi_L L + (1-\alpha)\pi_h L$$

3. If $p = \pi_h L$, only h type buys, and there is only one equilibrium. Firms earn zero profit. This is inefficient equilibrium because low-type stays uninsured.

The key difference from the world with complete information is that now as you increase price, you will not increase profit, because low-type consumers drop out of the market and you deal with risky pool of drivers. That's adverse selection.



$$p_{eq} = \alpha\pi_L L + (1-\alpha)\pi_h L$$